

AN $R=T$ THEOREM FOR CERTAIN ORTHOGONAL SHIMURA VARIETIES

HAO PENG AND DMITRI WHITMORE

ABSTRACT. We prove an almost minimal $R = T$ theorem for self-dual Galois representations with coefficients in a finite field satisfying a property called rigid. We also prove the rigidity property for a large family of residual Galois representations attached to regular algebraic self-dual representations. Our theorem is based on a Taylor–Wiles patching argument for G -valued Galois representation, where G equals GO_{2m} or GSp_{2m} .

CONTENTS

1. Introduction	1
1.1. Notation and conventions	3
1.2. Acknowledgments	5
2. Preliminaries	5
2.1. Orthogonal Satake parameters and orthogonal Hecke algebras	5
2.2. Automorphic representations	6
2.3. Functorial liftings	8
2.4. Almost unramified representations	8
3. Deformation	10
3.1. Some algebras	11
3.2. Deformation problems	11
3.3. Fontaine–Laffaille deformations	15
3.4. Tame representations	17
3.5. Minimally ramified representations	18
3.6. Level-raising deformations	22
4. An almost minimal $R=T$ theorem	23
4.1. Rigid Galois representations	23
4.2. Adequate subgroups	23
4.3. Finiteness of deformation rings	24
4.4. An almost minimal $R=T$ theorem	25
4.5. Local computations	27
4.6. Taylor–Wiles patching argument	29
5. Rigidity	33
5.1. Rigidity of odd symmetric powers of elliptic curves	33
5.2. Rigidity of automorphic Galois representations	34
References	36

1. INTRODUCTION

Let F be a totally real number field. We prove an $R = T$ theorem (see Theorem 4.4.2) for self-dual Galois representations $\bar{\rho}$ of Gal_F with coefficients in a finite field satisfying certain *rigidity* property (see Definition 4.1.1). We then show that for any elliptic curve A over F , the residual Galois representations attached to symmetric powers of A is rigid when the residual characteristic is large. More generally, we show that if Π is a regular algebraic symplectically self-dual cuspidal representation of general linear groups over F with a supercuspidal component, then the residual residual Galois representations attached to Π is rigid when the residual characteristic is large.

The $R = \mathbb{T}$ theorem for GL_2 are crucial to Wiles and Taylor–Wiles’ work on the Fermat’s Last Theorem; cf. [Wil95, TW95]. For the original Taylor–Wiles method to work when considering a representation

$$\rho : \text{Gal}(\overline{F}/F) \rightarrow G(\overline{\mathbb{Q}_\ell}),$$

valued in a (not necessarily connected) reductive group G , one needs

$$[F : \mathbb{Q}](\dim G - \dim B) = \sum_{v|\infty} H^0(\text{Gal}(\overline{F}_v/F_v), \text{ad}^0(\overline{\rho})),$$

where B denotes a Borel subgroup of G , and $\text{ad}^0(\overline{\rho})$ denotes the kernel from $\text{ad}(\overline{\rho})$ to its G -coinvariants. As F is totally real, this numerical coincidence holds for the disconnected group $\mathcal{G}_n$ (with identity component $GL_n \times GL_1$) defined in [CHT08]. This leads to $R = \mathbb{T}$ theorems for unitary Shimura sets, which are essential in the proofs of potential modularity theorem for conjugate self-dual Galois representations and of the Sato–Tate conjecture (cf. for example, [Tay08] and [HSBT10]). The case of unitary Shimura varieties is treated in [LTX+24].

This numerical coincidence also holds for $G = \text{GSp}(2m)$ or $G = \text{GO}(2m)$. The corresponding $R = \mathbb{T}$ theorems for definite classical groups are treated in the work of Xiaoyu Zhang [Zha25]. In this work, we consider orthogonal Shimura varieties (including orthogonal Shimura sets) associated to special orthogonal groups that are not necessarily unramified at every finite place, and with arbitrary neat level structures; this flexibility is crucial for our applications. For example, the main result will be used in the Ph.D. thesis of the first author [Pen26], which proves the Beilinson–Bloch–Kato conjecture for a large family of regular algebraic self-dual cuspidal representations of general linear groups over F ; see [Pen25b, §1.2].

We state one of the results that is the least technical to formulate. Let Π be a 0-REASDC (that is, regular essentially algebraic self-dual cuspidal with trivial central character) representation of $GL_{2m}(\mathbf{A}_F)$ for some positive integer m (see Definition 2.2.2). Let $E \subset \mathbb{C}$ be a number field such that there exists a family of Galois representations

$$\{\rho_{\Pi,\lambda} : \text{Gal}(\overline{F}/F) \rightarrow GL_{2m}(E_\lambda)\}_\lambda$$

indexed by finite places of E such that $\rho_{\Pi,\lambda}$ is conjugate to $\rho_{\Pi,\lambda}^\vee(1 - 2m)$ (see Proposition 2.2.3, Definition 2.2.5, and Remark 2.2.6). Let Σ be a finite set of finite places of F such that for every finite place v of F , Π_v is unramified and the underlying rational prime of v is unramified in F .

Consider a pair $(\mathbf{V}, \mathcal{K})$ in which

- $\mathbf{V}$ is a nondegenerate quadratic space over F of dimension $2m + 1$ that is of signature $(N, 0)$ or $(N - 2, 2)$ at every infinite place, and
- $\mathcal{K} = \prod_v \mathcal{K}_v$ is a neat open compact subgroup of $\text{SO}(\mathbf{V})(\mathbf{A}_F^\infty)$, such that $\mathcal{K}_v$ is the stabilizer of a self-dual lattice for every v not in Σ .

We assume that there exists a cuspidal automorphic representation π of $\text{SO}(\mathbf{V})(\mathbf{A}_F)$ with nonzero $\mathcal{K}$ -invariants and with automorphic functorial lift Π (see Definition 2.3.1). Let $\mathbb{T}^{\mathbf{m}, \Sigma \cup \Sigma(\ell_\lambda)}$ denote the Hecke algebra of $\text{SO}(\mathbf{V})$ away from Σ and places lying above ℓ (see Definition 2.1.2). Every such pair $(\mathbf{V}, \mathcal{K})$ defines a complex Shimura variety $\text{Sh}(\mathbf{V}, \mathcal{K})$ of dimension $d(\mathbf{V})$. The Archimedean weights ξ of Π (see Definition 2.2.2) defines an $\mathcal{O}_\lambda$ -linear local system $\mathcal{L}_{\xi,\lambda}$ on $\text{Sh}(\mathbf{V}, \mathcal{K})$. Let $\mathbb{T}_\lambda$ be the image of $\mathbb{T}^{\mathbf{m}, \Sigma \cup \Sigma(\ell_\lambda)}$ in $\text{End}_{\mathcal{O}_\lambda} \left(H_{\text{ét}}^{d(\mathbf{V})}(\text{Sh}(\mathbf{V}, \mathcal{K}), \mathcal{L}_{\xi,\lambda}) \right)$.

For every finite place λ of E , we write ℓ_λ its underlying rational prime and $\mathcal{O}_\lambda$ the ring of integers of E_λ .

Theorem A. *Suppose that*

- (1) *there exists a finite place v of F at which Π_v is supercuspidal, and*
- (2) *either the local system $\mathcal{L}_{\xi,\lambda}$ is constant or $d(\mathbf{V}) = 0$.*

Then for all but finitely many finite places λ of E , we can attach to Π a maximal ideal $\mathfrak{m}_\lambda$ of $\mathbb{T}^{\mathbf{m}, \Sigma \cup \Sigma(\ell_\lambda)}$ with residue field $\mathcal{O}_\lambda/\lambda$, such that

- (1) *the localization $\mathbb{T}_{\lambda, \mathfrak{m}_\lambda}$ is canonically isomorphic to a commutative $\mathcal{O}_\lambda$ -algebra $R_{\mathcal{J}_\lambda}$ that classifies self-dual deformations of the residual representation of $\rho_{\Pi,\lambda}$ that are crystalline with regular Fontaine–Laffaille weights at places above ℓ_λ and unramified outside Σ and places above ℓ_λ ; and*
- (2) *the localization $H_{\text{ét}}^{d(\mathbf{V})}(\text{Sh}(\mathbf{V}, \mathcal{K}), \mathcal{L}_{\xi,\lambda})_{\mathfrak{m}_\lambda}$ is a finite free module over $\mathbb{T}_{\lambda, \mathfrak{m}_\lambda}$.*

Remark. The finite set of finite places of E that are excluded in Theorem A can be effectively bounded.

Theorem A extends previous results in the conjugate self-dual setting [CHT08, LTX⁺24] to the self-dual setting. It is a consequence of the two main results of this article, namely, Theorem 4.4.2 and Theorem 5.2.3. In this article, a more general result is proved for orthogonal Shimura varieties of type B or D^R, under an extra assumption on certain vanishing of localized cohomologies of $\mathrm{Sh}(\mathbf{V}, \mathcal{K})$ off middle degree (see Theorem 4.4.2(D4)). Note that, when $\mathcal{L}_{\xi, \lambda}$ is a constant local system, this assumption is proved by the author [Pen25a] when $\mathrm{Sh}(\mathbf{V}, \mathcal{K})$ is proper, and later by Yang–Zhu [YZ25] in general.

The proof of Theorem A uses the Taylor–Wiles method following [Whi23], which provided a framework for handling Galois representations valued in arbitrary connected reductive groups under a mild condition on the residual image. Let $\bar{\rho}$ be the residual representation of $\rho_{\Pi, \lambda}$, considered as a representation valued in $G(\mathcal{O}_\lambda/\lambda)$, where $G = \mathrm{GSp}(2m)$ or $\mathrm{GO}(2m)$. In the Taylor–Wiles method, one considers liftings of $\bar{\rho}$ for which ramification is allowed at certain auxiliary sets of places. In our setting, a Taylor–Wiles place v of F satisfies

- $\bar{\rho}$ is unramified at v ;
- the residue field $\kappa(v)$ satisfies $\kappa(v) \equiv 1 \pmod{\ell}$; and
- the image of the Frobenius element $\bar{\rho}(\phi_v)$ is semisimple and contained in the identity component of G .

We then consider Taylor–Wiles deformation problems as defined in [Whi23], which consists of liftings of $\bar{\rho}$ for which the inertia group I_v is valued in certain torus. Since our deformation problem is in general not equal to the unrestricted deformation problem, some kind of local-global compatibility is required to show that Galois representations (attached to those automorphic representations arising in the Taylor–Wiles method) satisfy our Taylor–Wiles deformation problems. The main input here (aside from local-global compatibility up to semisimplification) is the local Langlands correspondence for principal series representations constructed in [Sol25], whose construction is explicit enough for us to establish the necessary vanishing of the monodromy operator in the associated Weil–Deligne representation whilst also coinciding with Arthur’s construction of local Langlands for classical groups.

In order to control the arising universal deformation rings, one requires the existence sets of Taylor–Wiles places for which a certain Galois cohomology group (the dual Selmer group) is forced to vanish. We impose a ‘big image’ condition on $\bar{\rho}$ using the notion of G -adequate subgroups from [Whi23], which can be verified to hold whenever ℓ (the rational prime underlying λ) is large and $\bar{\rho}$ is absolutely irreducible. Moreover, in §5, we verify that $\bar{\rho}$ is absolutely irreducible for all but finitely many λ .

Finally, in order to deal with finite places of F away from ℓ where $\bar{\rho}$ is ramified, we use the notion of *minimally ramified* representations from [Boo19a]. We introduce a notion of *rigid* Galois representation in §4.1, so that when $\bar{\rho}$ is rigid, any lifting of $\bar{\rho}$ is minimally ramified at finite places of F away from ℓ . Moreover, in §5, we verify that $\bar{\rho}$ is rigid for all but finitely many λ . We emphasize that the proof of rigidity ultimately relies on classical modularity lifting theorems, in particular, the classical $R = \mathbb{T}$ theorems in the unitary setting [CHT08, Tho12].

1.1. Notation and conventions.

Notation 1.1.1.

- Let $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ be the monoid of nonnegative integers and set $\mathbb{Z}_+ = \mathbb{N} \setminus \{0\}$. We write $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ for the integers, rational numbers, real numbers, and complex numbers, respectively.
- All rings are commutative and unital, and ring homomorphisms preserve units.
- We take square roots only of positive real numbers and always choose the positive root.
- Suppose X is a set.
 - Let $\mathbf{1} \in X$ denote the distinguished trivial element (this notation is only used when the notion of triviality is clear from context).
 - Let $\#X$ be the cardinality of X .
 - For $a, b \in X$, we define the Kronecker symbol

$$\delta_{a,b} := \begin{cases} 1 & \text{if } a = b \\ 0 & \text{if } a \neq b \end{cases}.$$

- For two topological (resp. algebraic) groups H and G , we write $H \leq G$ if H is a closed subgroup of G .

- For each algebraic group G over a field K , we write G° for the neutral component of G .
- For each square matrix M over a ring, we write $M^\top$ for its transpose.
- For each positive integer n , we denote by Sym_n the n -th symmetric group acting on $\{1, \dots, n\}$.
- We use standard notation from category theory. The category of sets is denoted by Set . The category of schemes is denoted by Sch .
- For each rational prime p , we fix an algebraic closure $\overline{\mathbb{Q}_p}$ of $\mathbb{Q}_p$ with residue field $\overline{\mathbb{F}_p}$. For every integer $r \in \mathbb{Z}_+$, we denote by $\mathbb{Q}_{p^r}$ the unique unramified extension of $\mathbb{Q}_p$ of degree r inside $\overline{\mathbb{Q}_p}$, and by $\mathbb{F}_{p^r}$ its residue field.
- We fix a totally real number field F and an algebraic closure $\overline{F}$ of F . Set $\text{Gal}_F := \text{Gal}(\overline{F}/F)$.
- We fix an integer $m \geq 1$ and an element $\mathfrak{d}$ in F such that $(-1)^m \mathfrak{d}$ is totally nonnegative (that is, $(-1)^m \tau(\mathfrak{d}) \geq 0$ for every embedding $\tau : F \rightarrow \mathbb{R}$). Denote by $\chi_{\mathfrak{d}} : F^\times \backslash \mathbf{A}_F^\times \rightarrow \mathbb{C}^\times$ the character corresponding to the Galois extension $F(\sqrt{\mathfrak{d}})/F$ via global class field theory (which is either trivial or quadratic).
- We denote by Mat_{2m} (resp. GL_{2m}) the scheme over $\mathbb{Z}$ of $2m \times 2m$ square matrices (resp. invertible $2m \times 2m$ square matrices).
- We define $N = 2m + \delta_{\mathfrak{d},0}$, and assume $N \geq 3$.
- For each positive integer k , we denote by I_k the unit element of $\text{GL}_N(\mathbb{Z})$. Let $J_k = \sum_{i=1}^k e_{1,k+1-i}$, which is an anti-diagonal $k \times k$ matrix in $\text{Mat}_k(\mathbb{Z})$, let $J'_k = \sum_{i=1}^k (-1)^{i-1} e_{i,k+1-i}$, which is an anti-diagonal $k \times k$ matrix in $\text{Mat}_k(\mathbb{Z})$, and let $N_k = \sum_{i=1}^{k-1} e_{i,i+1}$, which is a nilpotent matrix.
- Denote by Σ^∞ the set of real embeddings of F , and by Σ^{fin} the set of finite places of F .
- For any set S of rational primes, denote by $\Sigma(S)$ the set of all finite places v of F with $\text{char } \kappa_v \in S$. If $S = \{p\}$ is a singleton, we write simply $\Sigma(p) := \Sigma(\{p\}) = \{v \in \Sigma_F^{\text{fin}} : v|p\}$.
- Let Σ^{bad} denote the (finite) set of finite places of F whose underlying rational prime is ramified in $F(\sqrt{\mathfrak{d}})$.
- For each finite set Σ of places of F , we write

$$\mathbf{A}_{F,\Sigma} := \prod_{v \in \Sigma} F_v, \quad \mathbf{A}_F^\Sigma := \prod'_{v \in \Sigma_F \setminus \Sigma} F_v, \quad \mathbf{A}_F := \mathbf{A}_F^\infty \times (F \otimes_{\mathbb{Q}} \mathbb{R})$$

- For each place v of F , we
 - write $\mathcal{O}_v$ and F_v for the completion of $\mathcal{O}_F$ (resp. F) at v ;
 - let κ_v denote the residue field, $\|v\| := \#\kappa_v$, and write $\text{char } \kappa_v$ for the residue characteristic.
 - we fix an algebraic closure $\overline{F}_v$ of F_v with residue field $\overline{\kappa}_v$ and an embedding $\iota_v : \overline{F} \hookrightarrow \overline{F}_v$ extending $F \hookrightarrow F_v$; via ι_v we regard $\text{Gal}_{F_v} := \text{Gal}(\overline{F}_v/F_v)$ as a decomposition subgroup of Gal_F ;
 - for any map $r : \text{Gal}_F \rightarrow X$, we write $r_v := r|_{\text{Gal}_{F_v}}$;
 - let $I_{F_v} \subset \text{Gal}_{F_v}$ be the inertia subgroup;
 - let $W_{F_v} \subset \text{Gal}_{F_v}$ be the Weil group.
 - fix an *arithmetic* Frobenius element $\phi_v \in \text{Gal}_{F_v}$.
- For each finite set Σ of finite places of F , we denote by $\text{Gal}_{F,\Sigma}$ the Galois group of the maximal subextension of $\overline{F}/F$ that is unramified outside $\Sigma \cup \Sigma^\infty$.
- For each rational prime ℓ , we denote by $\varepsilon_\ell : \text{Gal}_F \rightarrow \mathbb{Z}_\ell^\times$ the ℓ -adic cyclotomic character.
- Suppose Γ, G are groups and L is a ring.
 - We denote by Γ^{ab} the maximal abelian quotient of Γ ;
 - For a homomorphism $\rho : \Gamma \rightarrow \text{GL}_r(L)$ for some $r \in \mathbb{Z}_+$, we denote by $\rho^\vee : \Gamma \rightarrow \text{GL}_r(L)$ the contragredient homomorphism, which is defined by the formula $\rho^\vee(x) = (\rho(x)^\top)^{-1}$.
 - We say that two homomorphisms $\rho_1, \rho_2 : \Gamma \rightarrow G$ are conjugate if there exists an element $g \in G$ such that $\rho_1 = g \circ \rho_2 \circ g^{-1}$.
 - Suppose Γ is a subgroup of another group $\tilde{\Gamma}$, and an element $\gamma \in \tilde{\Gamma}$ normalizes $\tilde{\Gamma}$. For any $\rho : \Gamma \rightarrow G$ for some $r \in \mathbb{Z}_+$, we let $\rho^\gamma : \Gamma \rightarrow G$ be the homomorphism defined by $\rho^\gamma(x) = \rho(\gamma x \gamma^{-1})$ for every $x \in \Gamma$.

- For each positive integer n , let μ_n be the group scheme over $\text{Spec } \mathbb{Z}$ representing the group of n -th roots of unity.

1.2. Acknowledgments. We thank Anne-Marie Aubert, Yifeng Liu, Zeyu Wang, Yichao Tian, Liang Xiao, and Jialiang Zou for helpful conversations.

2. PRELIMINARIES

2.1. Orthogonal Satake parameters and orthogonal Hecke algebras. In this subsection, we introduce quadratic spaces, their associated special orthogonal groups and orthogonal Hecke algebras.

Definition 2.1.1. Let R be a $\mathcal{O}_F$ -algebra. A quadratic space over R of dimension N is a projective R -module V of rank N together with a symmetric perfect pairing

$$V \times V \rightarrow R$$

that is R -linear in both variables.

For any quadratic space V of dimension N over R , we denote by $V_{\sharp}$ the quadratic space $V \oplus Re$ where $\|e\| = 1$, which is a quadratic space of dimension $N + 1$ over R . For any R -linear isometry $f : V \rightarrow V'$ of quadratic spaces over R , we denote by $f_{\sharp} : V_{\sharp} \rightarrow V'_{\sharp}$ the induced isometry.

We introduce abstract orthogonal Satake parameters.

Definition 2.1.2. Let L be a ring. For any multiset $\alpha = \{\alpha_1, \dots, \alpha_{2m}\} \subset L$, we put

$$P_{\alpha}(T) = \prod_{i \in [2m]_+} (T - \alpha_i) \in L[T].$$

For each finite place v of F not in Σ^{bad} , we define an *abstract orthogonal Satake parameter* in L at v to be a multiset α consisting of $2m$ elements in L such that $P_{\alpha}(T) = \chi_{\mathfrak{d}}(\phi_v) T^{2m} P_{\alpha}(T^{-1})$.

For each finite place v of F not in Σ^{bad} , let $\Lambda_v^{m, \mathfrak{d}}$ be the unique up to isomorphism quadratic space over $\mathcal{O}_v$ of rank N with discriminant $\mathfrak{d}$ (resp. discriminant 1) when N is even (resp. odd), and $\text{SO}_v^{m, \mathfrak{d}}$ the associated special orthogonal group over $\mathcal{O}_v$, considered as a subgroup of $\text{GL}(\Lambda_v^{m, \mathfrak{d}})$. In particular, we are considering odd special orthogonal groups when $\mathfrak{d} = 0$ (thus N is odd), and considering even special orthogonal groups when $\mathfrak{d} \neq 0$ (thus N is even). Under suitable basis, the associated quadratic form of $\Lambda_v^{m, \mathfrak{d}}$ is given by the matrix

$$\begin{cases} J_{2m+1} & \text{if } \mathfrak{d} = 0, \\ J_{2m} & \text{if } \mathfrak{d} \in F_v^2 \setminus \{0\}, \\ \begin{bmatrix} & & & J_{m-1} \\ & 1 & 0 & \\ & 0 & -\mathfrak{d} & \\ J_{m-1} & & & \end{bmatrix} & \text{if } \mathfrak{d} \notin F_v^2. \end{cases}$$

Consider the integral local spherical Hecke algebra

$$\mathbb{T}_v^{m, \mathfrak{d}} := \mathbb{Z}[\text{SO}_v^{m, \mathfrak{d}}(\mathcal{O}_{F_v}) \backslash \text{SO}_v^{m, \mathfrak{d}}(F_v) / \text{SO}_v^{m, \mathfrak{d}}(\mathcal{O}_{F_v})]$$

with unit element $\mathbf{1}_{\text{SO}_v^{m, \mathfrak{d}}(\mathcal{O}_{F_v})}$. We write $A_v^{m, \mathfrak{d}}$ for a maximal split diagonal torus of $\text{SO}_v^{m, \mathfrak{d}}$ and $X_{\bullet}(A_v^{m, \mathfrak{d}})$ its cocharacter group, then we have the Satake transform

$$(2.1) \quad \text{CT}^{\text{cl}} : \mathbb{Z} \left[\sqrt{\|v\|} \right] \otimes \mathbb{T}_v^{m, \mathfrak{d}} \rightarrow \mathbb{Z} \left[\sqrt{\|v\|} \right] [X_{\bullet}(A_v^{m, \mathfrak{d}})].$$

It is a homomorphism of rings.

Definition 2.1.3. For a finite set Σ of finite places of F containing Σ^{bad} , we define the *abstract orthogonal Hecke algebra* away from Σ to be the restricted tensor product

$$\mathbb{T}^{m, \mathfrak{d}, \Sigma} := \bigotimes'_{v \in \Sigma_F^{\text{fin}} \setminus \Sigma} \mathbb{T}_v^{m, \mathfrak{d}}$$

with respect to unit elements. It is a ring.

Definition 2.1.4. Suppose v is a finite place of F not in Σ^{bad} , and α is an abstract orthogonal Satake parameter in L at v . We construct a Hecke character of $\mathbb{T}_v^{m, \mathfrak{d}}$. There are two cases:

- If $\mathfrak{d}$ is a square in F_v , then a set of representatives of $X_\bullet(A_v^{m,\mathfrak{d}})$ can be taken as

$$\{(t_1, \dots, t_N) | t_1, \dots, t_N \in \mathbb{Z} \text{ satisfying } t_i + t_{N+1-i} = 0 \text{ for all } i \in [N]_+\}.$$

Because Π_v has trivial central character, we can choose an ordering of α as $(\alpha_1, \dots, \alpha_{2m})$ satisfying $\alpha_{N+1-i} = \alpha_i^{-1}$ for each $i \in [N]_+$. Then we have a unique ring homomorphism

$$\mathbb{Z} \left[\sqrt{\|v\|} \right] [X_\bullet(A_v^{m,\mathfrak{d}})] \rightarrow \mathbb{C}$$

sending $(t_1, \dots, t_N)$ to $\prod_{i \in [m]_+} \alpha_i^{t_i}$. Composing with the Satake transform (2.1), we obtain a ring homomorphism

$$\phi_{\Pi_v} : \mathbb{T}_v^{m,\mathfrak{d}} \rightarrow \mathbb{C}.$$

It is independent of the ordering of α .

- If $\mathfrak{d}$ is not a square in F_v , then $N = 2m$, and a set of representatives of $X_\bullet(A_v^{m,\mathfrak{d}})$ can be taken as $\{(t_1, \dots, t_{2m}) | t_1, \dots, t_N \in \mathbb{Z} \text{ satisfying } t_i + t_{2m+1-i} = 0 \text{ for all } i \in [2m]_+ \text{ And } t_m = t_{m+1} = 0\}$.

Because Π_v has central character $\chi_{\mathfrak{d},v}$, we can choose an ordering of α as $(\alpha_1, \dots, \alpha_{2m})$ satisfying $\alpha_m = 1, \alpha_{m+1} = -1$, and $\alpha_{2m+1-i} = \alpha_i^{-1}$ for each $i \in [m-1]_+$. Then we have a unique ring homomorphism

$$\mathbb{Z} \left[\sqrt{\|v\|} \right] [X_\bullet(A_v^{m,\mathfrak{d}})] \rightarrow \mathbb{C}.$$

sending $(t_1, \dots, t_{2m})$ to $\prod_{i \in [m-1]_+} \alpha_i^{t_i}$. Composing with the Satake transform (2.1), we obtain a ring homomorphism

$$\phi_{\Pi_v} : \mathbb{T}_v^{m,\mathfrak{d}} \rightarrow \mathbb{C}.$$

It is independent of the ordering of α .

2.2. Automorphic representations. In this subsection, we recall some facts concerning automorphic representations.

Definition 2.2.1. We denote by $\mathbb{Z}_{\leq}^{2m}$ the subset of $\mathbb{Z}^{2m}$ consisting of nondecreasing sequences. For a finite set T and an element $\xi = (\xi_\tau)_{\tau \in T} \in (\mathbb{Z}_{\leq}^{2m})^T$, set

$$a_\xi := \min_{\tau \in T} \{\xi_{\tau,1}\}, \quad b_\xi := \max_{\tau \in T} \{\xi_{\tau,2m}\} + N - 2$$

Definition 2.2.2. Suppose Π is an automorphic representation of $\mathrm{GL}_{2m}(\mathbf{A}_F)$. We say Π is $\mathfrak{d}$ -REASDC (that is, regular essentially algebraic self-dual cuspidal) if

- (1) Π is a cuspidal automorphic representation;
- (2) $\Pi \cong \Pi^\vee$;
- (3) the central character of Π is equal to $\chi_{\mathfrak{d}}$; and
- (4) for every Archimedean place τ of F , Π_τ has infinitesimal weight $\lambda_\tau = (\lambda_{\tau,1}, \dots, \lambda_{\tau,2m}) \in \mathbb{Z}^{2m} + \left(\left(\frac{N-1}{2} \right)^{(2m)} \right)$,¹ satisfying that $\lambda_{\tau,1} < \lambda_{\tau,2} < \dots < \lambda_{\tau,m} < 0 < \lambda_{\tau,m+1} < \lambda_{\tau,m+2} < \dots < \lambda_{\tau,2m}$.

If Π is $\mathfrak{d}$ -REASDC, then there is a unique element $\xi_\Pi = (\xi_{\tau,1}, \dots, \xi_{\tau,2m})_\tau \in (\mathbb{Z}_{\leq}^{2m})^{\Sigma^\infty}$, which we call the *Archimedean weights* of Π , satisfying $\xi_{\tau,i} = -\xi_{\tau,2m+1-i}$ for every τ and i , such that

$$\lambda_\tau = \xi_\tau + \left(1 - \frac{N}{2}, 2 - \frac{N}{2}, \dots, -\delta_{\mathfrak{d},0} - 1, -\delta_{\mathfrak{d},0}, \delta_{\mathfrak{d},0}, \delta_{\mathfrak{d},0} + 1, \dots, \frac{N}{2} - 1 \right).$$

Let v be a finite place of F . For every irreducible admissible representation Π of $\mathrm{GL}_{2m}(F_v)$, every rational prime ℓ , and every isomorphism $\iota_\ell : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}_\ell}$, we denote by $\mathrm{rec}_{2m}(\iota_\ell \Pi)$ the (Frobenius-semisimple) Weil–Deligne representation associated with $\iota_\ell \Pi$ via the local Langlands correspondence; see [Hen00, HT01]. For each representation ρ of Gal_{F_v} with coefficients in $\overline{\mathbb{Q}_\ell}$ that is continuous (resp. de Rham) if $v \in \Sigma(\ell)$ (resp. $v \notin \Sigma(\ell)$), let $\mathrm{WD}(\rho)$ be the corresponding Weil–Deligne representation of W_{F_v} (see, for example, [TY07, §1]).

¹This notation means an $2m$ -tuple with all entries being $\frac{N-1}{2}$.

Proposition 2.2.3. *Let Π be $\mathfrak{d}$ -REASDC representation of $\mathrm{GL}_{2m}(\mathbf{A}_F)$ with Archimedean weights ξ_Π . Let G denote the split smooth group scheme GSp_{2m} (resp. GO_{2m}) over $\mathbb{Q}$ if $\mathfrak{d} = 0$ (resp. $\mathfrak{d} \neq 0$). We denote by Std the standard representation of G over $\mathcal{O}$ of dimension $2m$, and by $\nu : G \rightarrow \mathrm{GL}_1$ the similitude character.*

(1) Π_v is tempered for each finite place of F .

(2) For any rational prime ℓ and any isomorphism $\iota_\ell : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}_\ell}$, there is a continuous homomorphism

$$\rho_{\Pi, \iota_\ell} : \mathrm{Gal}_F \rightarrow G(\overline{\mathbb{Q}_\ell})$$

unique up to conjugation, such that

$$(2.2) \quad \mathrm{WD}(\mathrm{Std} \circ \rho_{\Pi, \iota_\ell}|_{\mathrm{Gal}_{F_v}})^{\mathrm{F-ss}} = \iota_\ell \mathrm{rec}_{2m} \left(\Pi_v \otimes |\det|^{\frac{2-N}{2}} \right)$$

for every finite place v of F . Moreover, we have

$$\nu \circ \rho_{\Pi, \iota_\ell} = \varepsilon_\ell^{2-N}, \quad \det(\mathrm{Std} \circ \rho_{\Pi, \iota_\ell}) = \chi_\mathfrak{d} \cdot \varepsilon_\ell^{m(2-N)}.$$

(3) For every finite place v of F lying above ℓ , $\mathrm{Std} \circ \rho_{\Pi, \iota_\ell}|_{\mathrm{Gal}_{F_v}}$ is de Rham (and furthermore crystalline if Π_v is unramified) with regular Hodge–Tate weights contained in the range $[a_\xi, b_\xi]$.

Proof. We first note that $\Pi \otimes |\det|^{\frac{2-N}{2}}$ is regular algebraic. Then Part (1) follows from [Car12, Theorem 1.2]. For part (2): Let $J_\mathfrak{d} \in \mathrm{Mat}_N$ denote the N by N matrix

$$\begin{cases} J_{2m+1} & \text{if } \mathfrak{d} = 0, \\ \begin{bmatrix} & & & J_{m-1} \\ & 1 & 0 & \\ & 0 & -\mathfrak{d} & \\ J_{m-1} & & & \end{bmatrix} & \text{if } \mathfrak{d} \neq 0, \end{cases}$$

and let $\mathrm{SO}_N^\mathfrak{d}$ be the quasi-split special orthogonal group over F defined by

$$\mathrm{SO}_N^\mathfrak{d} = \{g \in \mathrm{GL}_N : g^\top J_\mathfrak{d} g = J_\mathfrak{d}\}.$$

Then $\mathrm{SO}_N^\mathfrak{d}$ is quasi-split at every place of F . It follows from Arthur’s multiplicity formula [Art13] that there is a automorphic representation π of $\mathrm{SO}_N^\mathfrak{d}(\mathbf{A}_F)$ whose functorial lifting to $\mathrm{GL}_{2m}(\mathbf{A}_F)$ is Π (cf. Definition 2.3.1). By [Shi24, Theorem 1.2.2], there is a continuous homomorphism

$$\rho_{\Pi, \iota_\ell} : \mathrm{Gal}_F \rightarrow G(\overline{\mathbb{Q}_\ell})$$

associated to π satisfying Equation (2.2) at every places of F where π is unramified. It follows from the Chebotarev density theorem that $\mathrm{Std} \circ \rho_{\Pi, \iota_\ell}$ is just the Galois representation associated to Π in the sense of [BLGGT14, Theorem 2.1.1]. Thus Equation (2.2) holds for each place of F , by [Car12, Theorem 1.1] and [Car14, Theorem 1.1]. The last two properties also follow from [Shi24] and the Chebotarev density theorem. Part (3) is proved in [CH13, Theorem 4.2]. $\square$

It is expected that these associated Galois representations form a compatible system defined over a number field (as defined in [BLGGT14, §5.1]). More explicitly, we define the notion of strong coefficient fields:

Definition 2.2.4. For each finite place v of F not in Σ^Π , let $\alpha(\Pi_v)$ denote the Satake parameter of Π_v , which is an abstract Satake parameter in $\mathbb{C}$ of dimension $2m$ (see Definition 2.1.2), and let $\mathbb{Q}(\Pi_v)$ denote the subfield of $\mathbb{C}$ generated by the coefficients of the polynomial

$$\prod_{\alpha \in \alpha(\Pi_v)} \left(T - \alpha \sqrt{\|w\|}^N \right) \in \mathbb{C}[T].$$

We define the *coefficient field* (also called the *Hecke field*) of Π to be the compositum of the fields $\mathbb{Q}(\Pi_v)$ for all finite places w of F not in Σ^Π , denoted by $\mathbb{Q}(\Pi)$.

Definition 2.2.5. Let Π be a $\mathfrak{d}$ -REASDC representation of $\mathrm{GL}_{2m}(\mathbf{A}_F)$ (see Definition 2.2.2) then a number field $E \subset \mathbb{C}$ is called a *strong coefficient field* for Π if it contains $\mathbb{Q}(\Pi)$ (see Definition 2.2.4), and for each finite place $\lambda \in \Sigma_E^{\mathrm{fin}}$, there exists a continuous homomorphism

$$\rho_{\Pi, \lambda} : \mathrm{Gal}_F \rightarrow \mathrm{GL}_{2m}(E_\lambda),$$

such that for any isomorphism $\iota_\ell : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}_\ell}$ inducing the place λ , $\rho_{\Pi, \lambda} \otimes_{E_{\lambda, \iota_\ell}} \overline{\mathbb{Q}_\ell}$ and $\text{Std} \circ \rho_{\Pi, \iota_\ell}$ are conjugate (see Proposition 2.2.3).

Remark 2.2.6. If Π is $\mathfrak{d}$ -REASDC representation of $\text{GL}_{2m}(\mathbf{A}_F)$, then a strong coefficient field of Π exists by the argument of [CH13, Proposition 3.2.5].

2.3. Functorial liftings. We recall we recall some general facts about the local and global functorial liftings for special orthogonal groups following [CZ21], [Ish24] and [Pen25a, §2]. For each element $\alpha \in \mathbb{C}^\times$, we denote by $\underline{\alpha} : K^\times \rightarrow \mathbb{C}^\times$ the unramified character with Satake parameter α .

Let K be a finite extension of $\mathbb{Q}_p$, and let V be a quadratic space over K . For an irreducible admissible representation π of $\text{SO}(V)$, we recall the local functorial liftings of π defined by endoscopy theory [Art13, CZ21, Ish24] in certain special cases.

Fix a minimal parabolic subgroup $P_{\min} \leq \text{SO}(V)$. By Langlands' classification, there exists a unique parabolic subgroup $P \leq G$ containing $P_{\min}$ with Levi quotient M , a unique tempered admissible irreducible representation τ of $M(K)$, and a strictly positive unramified character χ of $P(K)$ such that π is the unique irreducible quotient $J_P^G(\tau(\chi))$ of the normalized parabolic induction $I_P^G(\tau(\chi))$, called the *Langlands quotient*. Then we may write

$$M = G_0 \times \text{GL}_{r_1} \times \cdots \times \text{GL}_{r_t},$$

with G_0 the special orthogonal factor, under which

$$\chi = \mathbf{1} \boxtimes (\underline{\alpha}_1 \circ \det_{r_1}) \boxtimes \cdots \boxtimes (\underline{\alpha}_t \circ \det_{r_t})$$

for unique real numbers $1 < \alpha_1 < \cdots < \alpha_t$, where $\det_{r_i}$ denotes the determinant map on $\text{GL}_{r_i}(K)$. Suppose $\tau = \tau_0 \boxtimes \tau_1 \boxtimes \cdots \boxtimes \tau_t$ under the above decomposition, then $\text{FL}(\pi)$ is isomorphic to the unique irreducible quotient of the normalized parabolic induction

$$I_{P_{\text{GL}}}^{\text{GL}_{2m}} \left(\tau_t^\vee(\underline{\alpha}_t^{-1} \circ \det_{r_t}) \boxtimes \cdots \boxtimes \tau_1^\vee(\underline{\alpha}_1^{-1} \circ \det_{r_1}) \boxtimes \text{FL}(\tau_0) \boxtimes \tau_1(\underline{\alpha}_1 \circ \det_{r_1}) \boxtimes \cdots \boxtimes \tau_t(\underline{\alpha}_t \circ \det_{r_t}) \right),$$

where P_{GL} is a standard parabolic subgroup of GL_{2m} whose Levi quotient is $\text{GL}_{r_t} \times \cdots \times \text{GL}_{r_1} \times \text{GL}_{2m_0} \times \text{GL}_{r_1} \times \cdots \times \text{GL}_{r_t}$.

Now we consider the tempered admissible irreducible representation τ_0 . There exists a unique parabolic subgroup P_0 of G_0 containing $P_{\min} \cap G_0$ with Levi quotient group M_0 , and a discrete series representation μ of $M_0(K)$ such that τ_0 is a direct summand of the normalized parabolic induction $I_{P_0}^{G_0}(\mu)$. In fact, $I_{P_0}^{G_0}(\mu)$ is a direct sum of finitely many tempered representations of multiplicity one. Write $\mu = \mu_0 \boxtimes \mu_1 \boxtimes \cdots \boxtimes \mu_s$, similar to the previous case, where μ_0 is a discrete series representation of the special orthogonal factor of the Levi quotient of P_0 . Then under similar notation, $\text{FL}(\tau_0)$ is isomorphic to

$$I_{P_{\text{GL}}}^{\text{GL}_{2m_0}} (\mu_t^\vee \boxtimes \cdots \boxtimes \mu_1^\vee \boxtimes \text{FL}(\mu_0) \boxtimes \mu_1 \boxtimes \cdots \boxtimes \mu_t),$$

which is an irreducible admissible representation of $\text{GL}_{2m_0}(K)$. Here and henceforth, we suppress the parabolic subgroup P_{GL} of GL_{2m_0} when it is clear.

Finally, we recall the notion of automorphic functorial liftings.

Definition 2.3.1. Let $\mathbf{V}$ be a quadratic space over F of dimension N , and let π be a discrete automorphic representation of $\text{SO}(\mathbf{V})(\mathbf{A}_F)$. An *automorphic functorial lifting* of π is an automorphic representation $\text{FL}(\pi)$ of $\text{GL}_{2m}(\mathbf{A}_F)$ which is a finite isobaric sum of discrete automorphic representations of smaller general linear groups such that $\text{FL}(\pi)_v$ is a functorial lifting of π_v for all but finitely many unramified $v \in \Sigma_F^{\text{fin}}$.

By [CZ24, Theorem 2.1], [Ish24, Theorem 3.16], and the strong multiplicity one property for GL_N (see [PS79]), $\text{FL}(\pi)$ exists uniquely. Moreover, if $\text{FL}(\pi)$ is a finite isobaric sum of cuspidal self-dual representations of smaller general linear groups, then $\text{FL}(\pi)_v$ is a functorial lifting of π_v for every place v of F , by Arthur's multiplicity formula for generic A -packets; see [CZ24, Theorem 2.6] and [Ish24, Theorem 3.17].

2.4. Almost unramified representations. In this subsection, we study a special kind of representations of the nonsplit p -adic odd special orthogonal groups.

Let r be a positive integer. Let K be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_K$ and residue field κ . Let V be a quadratic space of dimension $2r + 1$ over K equipped with a lattice $\Lambda \subset V$ satisfying that $\Lambda \subset \Lambda^\vee$ and $\Lambda^\vee/\Lambda \cong \kappa$. Then $\text{SO}(V)$ is a nonsplit pure inner form of $\text{SO}_{2r+1, K}$. Let $\mathcal{K}$ be the stabilizer of Λ in SO'_{2r+1} , which is a special maximal subgroup of $\text{SO}(V)$.

Proposition 2.4.1. *Let π be an irreducible admissible representation of $\mathrm{SO}(V)(K)$ with nonzero invariants by $\mathcal{K}$. Then there exists an element $(\alpha_2, \dots, \alpha_r) \in (\mathbb{C}^\times)^{r-1}$ satisfying $1 \leq |\alpha_2| \leq \dots \leq |\alpha_r|$, unique up to permutation, such that the local functorial lifting $\mathrm{FL}(\pi)$ (see §2.3) is isomorphic to the unique irreducible quotient of the normalized parabolic induction*

$$\mathrm{I}_{P_{\mathrm{GL}}}^{\mathrm{GL}_{2r}} \left(\underline{\alpha_r^{-1}} \boxtimes \dots \boxtimes \underline{\alpha_2^{-1}} \boxtimes \mathrm{St}_2 \boxtimes \underline{\alpha_2} \boxtimes \dots \boxtimes \underline{\alpha_r} \right).$$

Here P_{GL} is a standard parabolic subgroup of GL_{2r} whose Levi quotient is $\mathrm{GL}_1^{r-1} \times \mathrm{GL}_2 \times \mathrm{GL}_1^{r-1}$, and St_2 is the Steinberg representation of $\mathrm{GL}_2(K)$.

Proof. We fix a decomposition

$$\Lambda = \mathcal{O}_{Ke_{-r}} \oplus \dots \oplus \mathcal{O}_{Ke_{-2}} \oplus \Lambda_3 \otimes \mathcal{O}_{Ke_2} \oplus \dots \oplus \mathcal{O}_{Ke_r},$$

in which $(e_i, e_j) = \delta_{i+j,0}$ for $2 \leq |i|, |j| \leq r$ and each e_i is orthogonal to Λ_3 . For each $1 \leq i \leq r$, set

$$V_{2i+1} := Ke_{-i} \oplus \dots \oplus Ke_{-2} \oplus \Lambda_3 \otimes_{\mathcal{O}_K} K \otimes Ke_2 \oplus \dots \oplus Ke_i,$$

which is a quadratic subspace of V . Let $P_{\min}$ be the minimal parabolic subgroup of $\mathrm{SO}(V)$ defined by the isotropic flag

$$Ke_{-r} \subset Ke_{-r} \oplus Ke_{-r+1} \subset \dots \subset Ke_{-r} \oplus \dots \oplus Ke_{-2},$$

with the Levi quotient

$$\mathrm{SO}(V_3) \times \mathrm{GL}(Ke_2) \times \dots \times \mathrm{GL}(Ke_r).$$

We now realize the relative Weyl group $W \cong \{\pm 1\}^{r-1} \rtimes \mathrm{Sym}_{r-1}$ explicitly as a subgroup of $\mathcal{K}$:

- We realize Sym_{r-1} as the stabilizer of the element $1 \in \{1, \dots, r\}$ in Sym_r ;
- for each $2 \leq i \leq r$, we let the $(i-1)$ -th “ -1 ” in W correspond to the element switching e_i and e_{-i} and fixing other e_j for $j \neq i$; and
- we let each $\sigma \in \mathrm{Sym}_{r-1}$ correspond to the element sending $e_{\pm i}$ to $e_{\pm \sigma(i)}$ for every $2 \leq i \leq r$, denoted by w_σ .

For each element $\alpha = (\alpha_2, \dots, \alpha_r) \in (\mathbb{C}^\times)^{r-1}$, let ϕ_α denote the function in $\mathrm{I}_{P_{\min}}^{\mathrm{SO}(V)}(\mathbf{1}_3 \boxtimes \underline{\alpha_2} \boxtimes \dots \boxtimes \underline{\alpha_r})$ that takes value 1 on $\mathcal{K}$, where $\mathbf{1}_3$ denotes the trivial representation of $\mathrm{SO}(V_3)(K)$.

Because π has nonzero invariants by $\mathcal{K}$, it is a constituent of an unramified principal series. In other words, there exists an element $\alpha = (\alpha_2, \dots, \alpha_r) \in (\mathbb{C}^\times)^{r-1}$ satisfying $1 \leq |\alpha_2| \leq \dots \leq |\alpha_r|$, unique up to permutation, such that π is a constituent of the normalized parabolic induction

$$\mathrm{I}_{P_{\min}}^{\mathrm{SO}(V)}(\mathbf{1}_3 \boxtimes \underline{\alpha_2} \boxtimes \dots \boxtimes \underline{\alpha_r}).$$

Note that there exists a unique nonnegative integer r_0 and a unique sequence of positive integers $r_1, \dots, r_t$ satisfying $r_0 + \dots + r_t = r$, such that $|\alpha_i| = 1$ for $2 \leq i \leq r_0$, and

$$1 < |\alpha_{r_0+1}| = \dots = |\alpha_{r_0+r_1}| < |\alpha_{r_0+r_1+1}| = \dots < |\alpha_{r_0+\dots+r_{t-1}+1}| = \dots = |\alpha_t|.$$

For each $1 \leq i \leq t$, let τ_i be the irreducible tempered unramified representation of $\mathrm{GL}_{r_i}(K)$ with Satake parameter

$$\left\{ \frac{\alpha_{r_0+\dots+r_{i-1}+1}}{|\alpha_{r_0+\dots+r_i}|}, \dots, \frac{\alpha_{r_0+\dots+r_i}}{|\alpha_{r_0+\dots+r_i}|} \right\}.$$

Let $P_{0,\min} := \mathrm{SO}(V_{2r_0+1}) \cap P_{\min}$ be a minimal parabolic subgroup of $\mathrm{SO}(V_{2r_0+1})$. As $\mathbf{1}_3 \boxtimes \underline{\alpha_2} \boxtimes \dots \boxtimes \underline{\alpha_{r_0}}$ is a discrete series representation of $P_{0,\min}(K)$, the normalized parabolic induction

$$\mathrm{I}_{P_{0,\min}}^{\mathrm{SO}(V_{2r_0+1})}(\mathbf{1}_3 \boxtimes \underline{\alpha_2} \boxtimes \dots \boxtimes \underline{\alpha_{r_0}})$$

is a finite direct sum of irreducible tempered representations of $\mathrm{SO}(V_{2r_0+1})(K)$. Let τ_0 be the unique direct summand with nonzero invariants under $\mathcal{K} \cap \mathrm{SO}(V_{2r_0+1})(K)$. In particular, we obtain a Langlands quotient

$$\mathrm{J} := \mathrm{J}_P^{\mathrm{SO}(V)} \left(\tau_0 \boxtimes \left(\boxtimes_{i=1}^t \tau_i \left(|\alpha_{r_0+\dots+r_i}| \circ \det_{r_i} \right) \right) \right),$$

where P is a parabolic subgroup of $\mathrm{SO}(V)$ containing P_0 whose Levi quotient is isomorphic to $\mathrm{SO}(V_{2r_0+1}) \times \mathrm{GL}_{r_1} \times \dots \times \mathrm{GL}_{r_t}$.

We claim that J has nonzero invariants under $\mathcal{K}$. Assuming this claim, then $\mathrm{FL}(\pi)$ is isomorphic to the unique irreducible quotient of

$$I_{P_{\mathrm{GL}}}^{\mathrm{GL}_{2r}} \left(\left(\boxtimes_{i=1}^t \tau_i^\vee \left(\underline{|\alpha_{r_0+\dots+r_i}^{-1}|} \circ \det_{r_i} \right) \right) \boxtimes \mathrm{FL}(\tau_0) \boxtimes \left(\boxtimes_{i=1}^t \tau_i \left(\underline{|\alpha_{r_0+\dots+r_i}|} \circ \det_{r_i} \right) \right) \right).$$

However, $\mathrm{FL}(\tau_0)$ is isomorphic to the normalized induction

$$I_{P_{\mathrm{GL}} \cap \mathrm{GL}_{2r_0}}^{\mathrm{GL}_{2r_0}} \left(\underline{\alpha_{r_0}^{-1}} \boxtimes \dots \boxtimes \underline{\alpha_2^{-1}} \boxtimes \mathrm{St}_2 \boxtimes \underline{\alpha_2} \boxtimes \dots \boxtimes \underline{\alpha_{r_0}} \right),$$

which is irreducible. So the assertion follows.

Now we prove the claim. Note that there is a canonical inclusion of $G(K)$ -representations

$$I_P^{\mathrm{SO}(V)} \left(\tau_0 \boxtimes \left(\boxtimes_{i=1}^t \tau_i \left(\underline{|\alpha_{r_0+\dots+r_i}|} \circ \det_{r_i} \right) \right) \right) \subset I_{P_{\min}}^{\mathrm{SO}(V)} (\mathbf{1}_3 \boxtimes \underline{\alpha_2} \boxtimes \dots \boxtimes \underline{\alpha_r}),$$

under which ϕ_α belongs to the former space by our choice of τ_0 . By [Kon03, Corollary 3.2], the claim is equivalent to

$$T_w \phi_\alpha \neq 0,$$

where $w \in W$ is the element acting trivially on V_{2r_0} and switching $e_{-(r_0+\dots+r_{i-1}+j)}$ with $e_{r_0+\dots+r_i+1-j}$ for every $1 \leq i \leq t$ and every $1 \leq j \leq r_i$; and T_w is the intertwining operator such that

$$T_w \phi_\alpha(g) = \int_{N(K)} \phi_\alpha(w^{-1}ng) dn, \quad g \in \mathrm{SO}(V)(K).$$

Here N is the nilpotent radical of P and the integral is absolutely convergent by [Wal03, Proposition IV.2.1].

By [Cas80, Theorem 3.1], we have $T_w \phi_\alpha = c(\alpha) \phi_{w\alpha}$, where $c(\alpha)$ is the Harish-Chandra c -function. Note that the regularity condition in [Cas80, §3] can be dropped by continuity of intertwining operators. Under suitable normalization of Haar measure, $c(\alpha)$ is given by the Gindikin–Karpelevich formula:

$$c(\alpha) = \prod_{i=r_0+1}^r \left(\frac{\alpha_i - \#\kappa^{-1}}{\alpha_i - 1} \prod_{\substack{1 \leq j \leq i-1 \\ |\alpha_j| < |\alpha_i|}} \frac{\alpha_i - \#\kappa^{-1}\alpha_j}{\alpha_i - \alpha_j} \prod_{j=1}^{i-1} \frac{\alpha_i \alpha_j - \#\kappa^{-1}}{\alpha_i \alpha_j - 1} \right).$$

In particular, $c(\alpha)$ is nonzero, and the claim follows. $\square$

Corollary 2.4.2. *Suppose π is a tempered irreducible admissible representation of $\mathrm{SO}(V)(K)$ with nonzero invariants under a special maximal compact open subgroup of $G(K)$, then the monodromy operator of $\mathrm{WD}(\mathrm{FL}(\pi))$ is conjugate to $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \oplus \mathbf{1}_{2m-2}$.*

Proof. It follows from Proposition 2.4.1 that the cuspidal support of π is of the form $((K^\times)^{m-1} \times \mathrm{SO}(V_3)(K), \chi \boxtimes \mathbf{1})$, where V_3 is the anisotropic quadratic space of rank 3 over K and χ is an unramified character of $(K^\times)^{m-1}$. We also know that χ is unitary because π is tempered. In particular, $\mathrm{FL}(\pi)$ is the (irreducible) parabolic induction from the Levi

$$\mathrm{GL}_1^{m-1} \times \mathrm{SO}(V_3)(K) \times \mathrm{GL}_1^{m-1}$$

with inducing data $\chi \boxtimes \mathrm{St}_2 \boxtimes \chi^{-1}$, where St_2 is the Steinberg representation of $\mathrm{GL}_2(K)$. The corollary follows immediately. $\square$

3. DEFORMATION

In this section, we work in the following setting.

Setup 3.0.1.

- Let ℓ be an odd rational prime.
- Let E be a subfield of $\overline{\mathbb{Q}_\ell}$ finite over $\mathbb{Q}_\ell$. We denote by $\mathcal{O}$ the ring of integers of E , by λ the maximal ideal, and by κ the residue field.
- Denote by $\mathcal{C}_\mathcal{O}$ the category of Noetherian complete local rings over $\mathcal{O}$ with residue field κ , with morphisms local homomorphisms inducing the identity on κ . Denote by $\mathcal{C}_\mathcal{O}^f$ the full subcategory of Artinian local rings over $\mathcal{O}$.
- For each object $R \in \mathcal{C}_\mathcal{O}$, we denote by $\mathfrak{m}_R$ the maximal ideal of R .

- Let G be the smooth group scheme GSp_{2m} (resp. GO_{2m}) over $\mathcal{O}$ if $\mathfrak{d} = 0$ (resp. $\mathfrak{d} \neq 0$). Here GSp_{2m} is defined by the standard alternating pairing on $\mathcal{O}^{\oplus(2m)}$ given by the matrix J'_{2m} , and GO_{2m} is defined by the standard symmetric pairing given by the matrix J_{2m} .
- We denote by $\mathfrak{g}$ the Lie algebra of G , by G° the identity component of G , by G' the derived subgroup of G° , by $\mathfrak{g}'$ the Lie algebra of G' , by Z_G the center of G .
- We denote by Std the standard representation of G over $\mathcal{O}$ of dimension $2m$, and by $\nu : G \rightarrow \mathrm{GL}_1$ the similitude character.
- For any smooth group scheme H over $\mathcal{O}$, we denote by $H^\wedge$ the formal completion of H along the unit section. In particular, for each object R in $\mathcal{C}_\mathcal{O}$, we have $H^\wedge(R) = \ker(H(R) \rightarrow H(\kappa))$.
- Let Γ be a profinite group satisfying the following ℓ -finiteness property: for any open subgroup $\Gamma_0 \leq \Gamma$, there are only finitely many continuous homomorphisms from Γ_0 to $\mathbb{Z}/\ell\mathbb{Z}$. (Note that this is true for the absolute Galois group of a local field and for the Galois group of the maximal extension of a number field unramified outside a finite set of places).

3.1. Some algebras. In this subsection, we carry out some algebra computations that will be used in the deformation theory.

Lemma 3.1.1. *Let R be an object of $\mathcal{C}_\mathcal{O}$ and let $\rho : \Gamma \rightarrow G(R)$ is a homomorphism. Suppose $\mathrm{Std} \circ \rho \pmod{\mathfrak{m}_R}$ is absolutely irreducible, then the centralizer of ρ in $(G')^\wedge(R)$ is trivial.*

Proof. The lemma is easily reduced to the case when R is Artinian. We then use induction on the length of R . The case of length one is immediate. In general we may choose an ideal I of R such that I has length one. By the induction hypothesis, the centralizer of ρ in $(G')^\wedge(R)$ lies in $G'(1 + I)$.

Then it suffices to show that $H^0(\Gamma, \mathrm{ad}^0(\rho) \otimes_R \kappa') = 0$ for any field κ' containing κ . But this follows from Schur's lemma and that ℓ is odd. $\square$

Lemma 3.1.2 (Carayol). *Let $R' \subset R$ be Noetherian complete local rings with maximal ideals $\mathfrak{m}_{R'}$ and $\mathfrak{m}_R$, respectively. Suppose that R' and R has the same residue field, and $\mathfrak{m}_R \cap R' = \mathfrak{m}_{R'}$. Let Γ be a profinite group and let $\rho : \Delta \rightarrow \mathrm{GL}_{2m}(R)$ is a continuous homomorphism, such that $\rho \pmod{\mathfrak{m}_R}$ is absolutely irreducible and that $\mathrm{tr}(\rho)(\Gamma) \subset R'$. Then there exists a $\mathrm{GL}_{2m}^\wedge(R)$ -conjugate ρ' of ρ valued in $\mathrm{GL}_{2m}(R')$.*

Proof. This is [CHT08, Lemma 2.1.10]; cf. [Car94, Théorème 2]. $\square$

We also need to the following version of Carayol's lemma for G -valued Galois representations.

Lemma 3.1.3. *Let $R' \subset R$ be Noetherian complete local rings with maximal ideals $\mathfrak{m}_{R'}$ and $\mathfrak{m}_R$, respectively. Suppose that R' and R has the same residue field κ with characteristic $\ell > 2$, and $\mathfrak{m}_R \cap R' = \mathfrak{m}_{R'}$. Let Γ be a profinite group and let $\rho : \Gamma \rightarrow G(R)$ be a continuous homomorphism, such that $\mathrm{Std} \circ \rho \pmod{\mathfrak{m}_R}$ is absolutely irreducible and that $\mathrm{tr}(\mathrm{Std} \circ \rho)(\Gamma) \subset R', \nu \circ \rho(\Gamma) \subset R'$. Then ρ is $(G')^\wedge(R)$ -conjugate to a homomorphism $\rho' : \Gamma \rightarrow G(R')$.*

Proof. It follows from Lemma 3.1.2 that $\mathrm{Std} \circ \rho$ is $\mathrm{GL}_{2m}^\wedge(R)$ -conjugate to a representation ρ' valued in $\mathrm{GL}_{2m}(R')$. Because ρ' and $\rho'^\top \otimes (\nu \circ \rho)$ are both valued in $\mathrm{GL}_{2m}(R')$ and they are conjugate in $\mathrm{GL}_{2m}(R)$, they are also conjugate in $\mathrm{GL}_{2m}(R')$ by [Car94, Théorème 1]. Suppose B is an element of $\mathrm{GL}_{2m}(R')$ such that $\rho' = B\rho'^\top \otimes (\nu \circ \rho)B^{-1}$. Then it follows from Schur's lemma that B is symmetric (resp. skew-symmetric) if N is even (resp. N is odd), and B defines a general orthogonal (resp. general symplectic) group isomorphic to G . Thus we see that ρ is $\mathrm{GL}_{2m}(R)$ -conjugate to a representation valued in $G(R')$. It is also easy to check that one may choose the conjugating matrix to be in $\mathrm{GL}_{2m}^\wedge(R)$. Finally, it follows from Schur's lemma that the conjugating matrix is also in $G(R)$. After rescaling, we may choose the conjugating matrix to be in $(G')^\wedge(R)$. $\square$

3.2. Deformation problems. In this subsection, we recall the notion of deformation problems. We work in the following setting.

Setup 3.2.1. Suppose $(\Gamma, \bar{\rho}, \chi)$ is a triple in which

- Γ is our fixed topological group,
- $\bar{\rho} : \Gamma \rightarrow G(\kappa)$ is a homomorphism, and
- $\chi : \Gamma \rightarrow \mathcal{O}^\times$ is a continuous homomorphism with reduction $\bar{\chi} : \Gamma \rightarrow \kappa^\times$,

such that $\nu \circ \bar{\rho} = \bar{\chi}$. Let $\text{ad}(\bar{\rho})$ (respectively, $\text{ad}^0(\bar{\rho})$) denote the representation of Γ on $\mathfrak{g}_\kappa$ (respectively, $\mathfrak{g}'_\kappa$) via the adjoint representation.

Definition 3.2.2. Let $f : R_1 \rightarrow R_0$ be a morphism in $\mathcal{C}_\mathcal{O}$ and $\rho_0 : \Gamma \rightarrow G(R_0)$ a continuous homomorphism. A *lifting* of ρ_0 to R_1 (with respect to χ) is a continuous homomorphism $\rho_1 : \Gamma \rightarrow G(R_1)$ such that $f_*(\rho_1) = \rho_0$ and $\nu_{R_1} \circ \rho_1 = \chi$. Let

$$D_{\bar{\rho}, \mathcal{O}}^\square : \mathcal{C}_\mathcal{O} \rightarrow \text{Set}$$

be the functor sending each object $R \in \mathcal{C}_\mathcal{O}$ to the set of liftings of $\bar{\rho}$ to R .

For each $R \in \mathcal{C}_\mathcal{O}$, two liftings ρ, ρ' of $\bar{\rho}$ are called *equivalent* if they are conjugate by an element of $(G')^\wedge(R)$. A *deformation* of $\bar{\rho}$ to R is defined to be an equivalence class of deformations of $\bar{\rho}$ to R . Let

$$D_{\bar{\rho}, \mathcal{O}} : \mathcal{C}_\mathcal{O} \rightarrow \text{Set}$$

be the functor sending each object $R \in \mathcal{C}_\mathcal{O}$ to the set of deformations of $\bar{\rho}$ to R .

$D_{\bar{\rho}, \mathcal{O}}^\square$ is representable by an element $R_{\bar{\rho}}^\square \in \mathcal{C}_\mathcal{O}$, called the *universal lifting ring*; see [Bal12, Theorem 1.2.2].

The tangent space of $D_{\bar{\rho}, \mathcal{O}}$ is naturally identified with $H^1(\Gamma, \text{ad}^0(\bar{\rho}))$.

Definition 3.2.3. A *local lifting condition* of $\bar{\rho}$ (with respect to χ) is a closed formal subscheme $\mathcal{D}^\square$ of $\text{Spf } R_{\bar{\rho}}^\square$ satisfying that $\mathcal{D}^\square(R)$ is closed under equivalence for every $R \in \mathcal{C}_\mathcal{O}$.

For each local lifting condition $\mathcal{D}^\square$, we denote by $\mathcal{I}^\square \subset R_{\bar{\rho}}^\square$ the closed ideal defining $\mathcal{D}^\square$. Note that $\mathcal{D}^\square$ naturally induces a *local deformation problem* denoted by $\mathcal{D}$, which is a subfunctor of $D_{\bar{\rho}, \mathcal{O}}$.

Definition 3.2.4. For each *local lifting condition* $\mathcal{D}^\square$ of $\bar{\rho}$ (with respect to χ) with the associated local deformation problem $\mathcal{D}$, there is an isomorphism

$$\text{Hom}_\kappa(\mathfrak{m}_{R_{\bar{\rho}}^\square}/(\mathfrak{m}_{R_{\bar{\rho}}^\square}^2, \lambda), \kappa) \cong \text{Hom}_{\mathcal{C}_\mathcal{O}}(R_{\bar{\rho}}^\square, \kappa[\varepsilon]/(\varepsilon^2)) \cong Z^1(\Gamma, \text{ad}^0(\bar{\rho})),$$

where $Z^1(\Gamma, \text{ad}^0(\bar{\rho}))$ denotes the group of 1-cocycles of Γ with values in the adjoint representation $\text{ad}^0(\bar{\rho})$ of Γ on $\mathfrak{g}'_\kappa$.

We define the *tangent space* of $\mathcal{D}$, denoted by $L(\mathcal{D})$, to be the image of the subspace

$$L^1(\mathcal{D}^\square) := \text{Hom}_\kappa(\mathfrak{m}_{R_{\bar{\rho}}^\square}/(\mathfrak{m}_{R_{\bar{\rho}}^\square}^2, \mathcal{I}^\square, \lambda), \kappa) \subset Z^1(\Gamma, \text{ad}^0(\bar{\rho}))$$

under the natural map $Z^1(\Gamma, \text{ad}^0(\bar{\rho})) \rightarrow H^1(\Gamma, \text{ad}^0(\bar{\rho}))$. Here $\mathcal{I}^\square \subset R_{\bar{\rho}}^\square$ is the closed ideal defining $\mathcal{D}^\square$.

Because $\mathcal{I}^\square$ is invariant under conjugation by $(G')^\wedge(R)$, we see that $L^1(\mathcal{D}^\square)$ is the preimage of $L(\mathcal{D})$ in $Z^1(\Gamma, \text{ad}^0(\bar{\rho}))$. In particular,

$$(3.1) \quad \dim_\kappa L^1(\mathcal{D}^\square) - \dim_\kappa L(\mathcal{D}) = \dim G'_\kappa - \dim_\kappa H^0(\Gamma, \text{ad}^0(\bar{\rho})).$$

Definition 3.2.5. Suppose $\Gamma = \text{Gal}_F$. A *global deformation problem* of $\bar{\rho}$ (with respect to χ) is a tuple $(\bar{\rho}, \chi, \Sigma, \{\mathcal{D}_v^\square\}_{v \in \Sigma})$ in which

- Σ is a finite set of finite places of F containing $\Sigma(\ell)$ and all places v such that $\bar{\rho}_v$ is ramified.
- $\mathcal{D}_v^\square$ is a local lifting condition of $\bar{\rho}$ with respect to χ_v for each $v \in \Sigma$.

For the rest of this subsection, we let $\Gamma = \text{Gal}_F$ and fix a global deformation problem $\mathcal{S} = (\bar{\rho}, \chi, \Sigma, \{\mathcal{D}_v^\square\}_{v \in \Sigma})$. For each subset $\Xi \subset \Sigma$, set

$$R_{\mathcal{S}, \Xi}^\square := \widehat{\bigotimes_{v \in \Xi} (R_{\bar{\rho}_v}^\square / \mathcal{I}_v^\square)},$$

where the tensor product is taken over $\mathcal{O}$. We recall from [CHT08, Definition 2.2.1] the notion of framed liftings and framed deformations.

Definition 3.2.6. For each subset $\Xi \subset \Sigma$, a Ξ -*framed lifting* of $\bar{\rho}$ to an object $R \in \mathcal{C}_\mathcal{O}$ is a tuple $(\rho; \{\beta_v\}_{v \in \Xi})$, in which ρ is a lifting of $\bar{\rho}$ to R with respect to χ (see Definition 3.2.2), and $\beta_v \in (G')^\wedge(R)$ for every $v \in \Xi$. We say that a Ξ -framed lifting $(\rho, \{\beta_v\}_{v \in \Xi})$ is of type $\mathcal{S}$ if ρ is unramified outside Σ and $\rho_v \in \mathcal{D}_v^\square(R)$ for every $v \in \Sigma$.

Two Ξ -framed liftings $(\rho; \{\beta_v\}_{v \in \Xi})$ and $(\rho'; \{\beta'_v\}_{v \in \Xi})$ are said to be equivalent if there exists $g \in (G')^\wedge(R)$ such that $\rho' = g \circ \rho \circ g^{-1}$ and $\beta'_v = g\beta_v$ for every $v \in \Xi$. A Ξ -*framed deformation* of $\bar{\rho}$ is an equivalence class of Ξ -framed liftings of $\bar{\rho}$.

If a Ξ -framed lifting is of type $\mathcal{S}$, so is any equivalent Ξ -framed lifting. We say that a Ξ -framed deformation is of type $\mathcal{S}$ if some (or equivalently, every) element is of type $\mathcal{S}$. Let

$$\text{Def}_{\mathcal{S}}^{\square\Xi} : \mathcal{C}_{\mathcal{O}} \rightarrow \text{Set}$$

be the functor sending each object $R \in \mathcal{C}_{\mathcal{O}}$ to the set of Ξ -framed deformations of $\bar{\rho}$ to R of type $\mathcal{S}$.

Recall that $\text{Gal}_{F,\Sigma}$ is the Galois group of the maximal subextension of $\bar{F}/F$ that is unramified outside $\Sigma \cup \Sigma^{\infty}$. Following [CHT08, p. 21], for each integer $i \in \mathbb{N}$, we denote by $H_{\mathcal{S},\Xi}^i(\text{Gal}_{F,\Sigma}, \text{ad}^0(\bar{\rho}))$ the cohomology of the complex

$$C_{\mathcal{S},\Xi}^i(\text{Gal}_{F,\Sigma}, \text{ad}^0(\bar{\rho})) := C^i(\text{Gal}_{F,\Sigma}, \text{ad}^0(\bar{\rho})) \oplus \bigoplus_{v \in \Sigma} \frac{C^{i-1}(F_v, \text{ad}^0(\bar{\rho}))}{M_v^{i-1}},$$

where $M_v^{i-1} = 0$ unless $v \in \Sigma \setminus \Xi$ and $i = 0$ in which case

$$M_v^i = \begin{cases} C^0(F_v, \text{ad}^0(\bar{\rho})) & \text{if } i = 0 \\ L^1(\mathcal{D}_v^{\square}) & \text{if } i = 1 \end{cases}.$$

The coboundary map is

$$C_{\mathcal{S},\Xi}^i(\text{Gal}_{F,\Sigma}, \text{ad}^0(\bar{\rho})) \rightarrow C_{\mathcal{S},\Xi}^{i+1}(\text{Gal}_{F,\Sigma}, \text{ad}^0(\bar{\rho})) : (\phi, (\psi_v)_{v \in \Sigma}) \mapsto (\partial\phi, (\phi|_{F_v} - \partial\psi_v)_{v \in \Sigma}).$$

Then there is an exact sequence

$$\begin{aligned} 0 &\rightarrow H_{\mathcal{S},\Xi}^0(\text{Gal}_{F,\Sigma}, \text{ad}^0(\bar{\rho})) \rightarrow H^0(\text{Gal}_{F,\Sigma}, \text{ad}^0(\bar{\rho})) \rightarrow \bigoplus_{v \in \Xi} H^0(F_v, \text{ad}^0(\bar{\rho})) \\ &\rightarrow H_{\mathcal{S},\Xi}^1(\text{Gal}_{F,\Sigma}, \text{ad}^0(\bar{\rho})) \rightarrow H^1(\text{Gal}_{F,\Sigma}, \text{ad}^0(\bar{\rho})) \rightarrow \left(\bigoplus_{v \in \Sigma \setminus \Xi} \frac{H^1(F_v, \text{ad}^0(\bar{\rho}))}{L(\mathcal{D}_v)} \right) \oplus \bigoplus_{v \in \Xi} H^1(F_v, \text{ad}^0(\bar{\rho})) \\ &\rightarrow H_{\mathcal{S},\Xi}^2(\text{Gal}_{F,\Sigma}, \text{ad}^0(\bar{\rho})) \rightarrow H^2(\text{Gal}_{F,\Sigma}, \text{ad}^0(\bar{\rho})) \rightarrow \bigoplus_{v \in \Sigma} H^2(F_v, \text{ad}^0(\bar{\rho})) \\ &\rightarrow H_{\mathcal{S},\Xi}^3(\text{Gal}_{F,\Sigma}, \text{ad}^0(\bar{\rho})) \rightarrow H^3(\text{Gal}_{F,\Sigma}, \text{ad}^0(\bar{\rho})) \rightarrow \dots \end{aligned}$$

Let $L(\mathcal{D}_v)^{\perp} \subset H^1(F_v, \text{ad}^0(\bar{\rho})(1))$ be the annihilator of $L(\mathcal{D}_v)$ under the local Tate duality induced from the Galois invariant perfect pairing

$$\text{ad}^0(\bar{\rho}) \times \text{ad}^0(\bar{\rho}) \rightarrow \kappa : (x, y) \mapsto \text{tr}(xy).$$

We define

$$(3.2) \quad H_{\mathcal{S}^{\perp},\Xi}^1(\text{Gal}_{F,\Sigma}, \text{ad}^0(\bar{\rho})(1)) := \ker \left(H^1(\text{Gal}_{F,\Sigma}, \text{ad}^0(\bar{\rho})(1)) \rightarrow \prod_{v \in \Sigma \setminus \Xi} \frac{H^1(F_v, \text{ad}^0(\bar{\rho})(1))}{L(\mathcal{D}_v)^{\perp}} \right).$$

Lemma 3.2.7. *The following statements hold:*

- (1) $H_{\mathcal{S},\Xi}^i(\text{Gal}_{F,\Sigma}, \text{ad}^0(\bar{\rho}))$ vanishes for $i > 3$.
- (2) $H_{\mathcal{S},\Xi}^0(\text{Gal}_{F,\Sigma}, \text{ad}^0(\bar{\rho})) = H^0(\text{Gal}_{F,\Sigma}, \text{ad}^0(\bar{\rho}))$ if $\Xi = \emptyset$ and vanishes if $\Xi \neq \emptyset$.
- (3) $\dim_{\kappa} H_{\mathcal{S},\Xi}^3(\text{Gal}_{F,\Sigma}, \text{ad}^0(\bar{\rho})) = \dim_{\kappa} H^0(\text{Gal}_{F,\Sigma}, \text{ad}^0(\bar{\rho})(1))$.
- (4) $\dim_{\kappa} H_{\mathcal{S},\Xi}^2(\text{Gal}_{F,\Sigma}, \text{ad}^0(\bar{\rho})) = \dim_{\kappa} H_{\mathcal{S}^{\perp},\Xi}^1(\text{Gal}_{F,\Sigma}, \text{ad}^0(\bar{\rho})(1))$.
- (5) (Wiles' formula)

$$\begin{aligned} \dim_{\kappa} H_{\mathcal{S},\Xi}^1(\text{Gal}_{F,\Sigma}, \text{ad}^0(\bar{\rho})) &= \delta_{\Xi,\emptyset} \cdot \dim_{\kappa} H^0(\text{Gal}_{F,\Sigma}, \text{ad}^0(\bar{\rho})) + \dim_{\kappa} H_{\mathcal{S}^{\perp},\Xi}^1(\text{Gal}_{F,\Sigma}, \text{ad}^0(\bar{\rho})(1)) \\ &\quad - \dim_{\kappa} H^0(\text{Gal}_{F,\Sigma}, \text{ad}^0(\bar{\rho})(1)) - [F : \mathbb{Q}](\dim G_{\kappa} - \dim B_{\kappa}) \\ &\quad + \sum_{v \in \Sigma \setminus \Xi} (\dim_{\kappa} L(\mathcal{D}_v) - \dim_v H^0(F_v, \text{ad}^0(\bar{\rho}))). \end{aligned}$$

Proof. Using the ℓ -finiteness property, the proof is the same as that of [CHT08, Lemma 2.3.4]. Here we use that $H^0(\text{Gal}_{F_v}, \text{ad}^0(\mathfrak{g})) = m(m + \delta_{\mathfrak{d},0} - 1) = (\dim G_{\kappa} - \dim B_{\kappa})$ for every real place v of F . This formula is a corollary of the following claim: Let $\langle -, - \rangle$ be the nondegenerate symmetric (resp. alternating) pairing

on $\kappa^{\oplus 2m}$ defined by J_{2m} (resp. J'_{2m}) when $N = 2m$ (resp. $N = 2m + 1$). (In particular, G_κ is the subgroup of elements of GL_{2m} that fix $\langle -, - \rangle$ up to a scalar.) Then $\bar{\rho}(\mathbf{c}_v)$ is $G(\kappa)$ -conjugate to the matrix

$$\begin{cases} \begin{bmatrix} \mathbf{1}_m & \\ & -\mathbf{1}_m \end{bmatrix} & \mathfrak{d} = 0, \\ J_{2m} & \mathfrak{d} \neq 0, \end{cases}$$

for any complex conjugation $\mathbf{c}_v$ at v .

We now prove this claim: We write $s = \rho_{\Pi, \iota_\ell}(\mathbf{c}_v)$. We first treat the case when $N = 2m + 1$. It follows from Proposition 2.2.3(2) that $\nu(s) = -1$. As $s^2 = 1$, s has eigenvalues ± 1 . Let $V^\pm$ be the eigenspace of ± 1 of s . Then

$$\langle v, w \rangle = \langle sv, sw \rangle = -\langle v, w \rangle$$

for $v, w \in V^+$. Thus V^+ is totally isotropic. The same argument shows that V^- is totally isotropic. Thus we can take a basis $\{x_1, \dots, x_m\}$ of V^+ and a basis $\{y_1, \dots, y_m\}$ of V^- , such that $\langle x_i, y_i \rangle = \delta_{i,j}$. As a consequence, in the basis $\{x_1, \dots, x_m, y_1, \dots, y_m\}$, s is represented by the matrix

$$\begin{bmatrix} \mathbf{1}_m & \\ & -\mathbf{1}_m \end{bmatrix}.$$

The claim follows.

In the setting when $N = 2m$, we have $\nu(s) = 1$. Let $V^\pm$ be the eigenspace of ± 1 of s . Then V^+ is orthogonal to V^- . We also know that $\dim V^+ = \dim V^- = m$ by [CLH16, Theorem 1.1]. The claim then follows easily. $\square$

Proposition 3.2.8. *Assume that $\mathrm{Std} \circ \bar{\rho}$ is absolutely irreducible. Then for each subset $\Xi \subset \Sigma$, the functor $\mathrm{Def}_{\mathcal{S}}^{\square_\Xi}$ is represented by an object $\mathbf{R}_{\mathcal{S}}^{\square_\Xi} \in \mathcal{C}_{\mathcal{O}}$. Set $\mathcal{R}_{\mathcal{S}} := \mathbf{R}_{\mathcal{S}}^{\square_\emptyset}$, called the global deformation ring (with respect to $\mathcal{S}$). Note that the natural transformation*

$$\mathrm{Def}_{\mathcal{S}}^{\square_\Xi} \rightarrow \prod_{v \in \Xi} \mathbf{D}_{\bar{\rho}_v, \mathcal{O}}^{\square} : (\rho, \{\beta_v\}_{v \in \Xi}) \mapsto (\beta_v^{-1} \rho_v \beta_v)_{v \in \Xi}$$

induces a tautological morphism $\mathbf{R}_{\mathcal{S}, \Xi}^{\square} \rightarrow \mathbf{R}_{\mathcal{S}}^{\square_\Xi}$ in $\mathcal{C}_{\mathcal{O}}$.

(1) *There is a canonical isomorphism*

$$\mathrm{Hom}_\kappa(\mathbf{m}_{\mathbf{R}_{\mathcal{S}, \Xi}^{\square}} / (\mathbf{m}_{\mathbf{R}_{\mathcal{S}, \Xi}^{\square}}^2, \lambda, \mathbf{m}_{\mathbf{R}_{\mathcal{S}, \Xi}^{\square}}), \kappa) \cong \mathrm{H}_{\mathcal{S}, \Xi}^1(\Gamma_{F, \Sigma}, \mathrm{ad}^0(\bar{\rho})),$$

where we regard $\mathbf{m}_{\mathbf{R}_{\mathcal{S}, \Xi}^{\square}}$ as its image under the tautological homomorphism $\mathbf{R}_{\mathcal{S}, \Xi}^{\square} \rightarrow \mathbf{R}_{\mathcal{S}}^{\square_\Xi}$.

(2) *If $\mathrm{H}_{\mathcal{S}, \Xi}^2(\Gamma_{F, \Sigma}, \mathrm{ad}^0(\bar{\rho})) = 0$ and $\mathcal{D}_v$ is formally smooth over $\mathcal{O}$ for each $v \in \Sigma \setminus \Xi$, then $\mathbf{R}_{\mathcal{S}}^{\square_\Xi}$ is a power series ring over $\mathbf{R}_{\mathcal{S}, \Xi}^{\square}$ in $\dim_\kappa \mathrm{H}_{\mathcal{S}, \Xi}^1(\Gamma_{F, \Sigma}, \mathrm{ad}^0(\bar{\rho}))$ variables.*

(3) *Choosing a universal lifting $\rho_{\mathcal{S}} : \Gamma_F \rightarrow G(\mathbf{R}_{\mathcal{S}})$ of $\bar{\rho}$ to $\mathbf{R}_{\mathcal{S}}$ determines an extension of the tautological homomorphism $\mathbf{R}_{\mathcal{S}} \rightarrow \mathbf{R}_{\mathcal{S}}^{\square_\emptyset}$ to an isomorphism*

$$\mathbf{R}_{\mathcal{S}} \widehat{\otimes}_{\mathcal{O}} \mathcal{T}_\Xi \cong \mathbf{R}_{\mathcal{S}}^{\square_\Xi},$$

where $\mathcal{T}_\Xi$ is the coordinate ring of the formal smooth group $\prod_{v \in \Xi} (G')^\wedge$ over $\mathcal{O}$. Note that $\mathcal{T}_\Xi$ is isomorphic to a formal power series algebra over $\mathcal{O}$ in $\#\Xi \cdot \dim G'_\kappa$ variables.

Proof. Note that $\mathrm{H}^0(\mathrm{Gal}_F, \mathrm{ad}^0(\bar{\rho})) = 0$ since $\mathrm{Std} \circ \bar{\rho}$ is absolutely irreducible. Thus the representability of $\mathbf{R}_{\mathcal{S}}^{\square_\Xi}$ can be proved the in the same way as the proof of [Pat16, Proposition 9.2(2)].

For part (1), recall that

$$\mathrm{Hom}_\kappa(\mathbf{m}_{\mathbf{R}_{\mathcal{S}, \Xi}^{\square}} / (\mathbf{m}_{\mathbf{R}_{\mathcal{S}, \Xi}^{\square}}^2, \lambda, \mathbf{m}_{\mathbf{R}_{\mathcal{S}, \Xi}^{\square}}), \kappa) \cong \mathrm{Hom}(\mathbf{R}_{\mathcal{S}}^{\square_\Xi} / (\mathbf{m}_{\mathbf{R}_{\mathcal{S}, \Xi}^{\square}}, \kappa[\varepsilon] / (\varepsilon^2))$$

is isomorphic to the subspace of $\mathrm{Def}_{\mathcal{S}}^{\square_\Xi}$ consisting of elements giving trivial liftings of $\bar{\rho}_v$ for $v \in \Xi$. Any Ξ -framed lifting of $\bar{\rho}$ is of the form

$$((1 + \phi\varepsilon)\bar{\rho}; \{\mathbf{1} + a_v\varepsilon\}_{v \in \Xi})$$

with $\phi \in Z^1(\text{Gal}_F, \text{ad}^0(\bar{\rho}))$. It is of type $\mathcal{S}$ if and only if $\phi|_{\text{Gal}_{F_v}} \in L^1(\mathcal{D}_v^\square)$ for every $v \in \Sigma$. It gives rise to a trivial lifting of $\bar{\rho}|_{\text{Gal}_{F_v}}$ if and only if

$$(\mathbf{1} - a_v \varepsilon)(\mathbf{1} + \phi|_{\text{Gal}_{F_v}} \varepsilon) \bar{\rho}(\mathbf{1} + a_v \varepsilon) = \bar{\rho}_v,$$

or equivalently, $\phi|_{\text{Gal}_{F_v}} = (1 - \text{ad}^0(\bar{\rho})|_{\text{Gal}_{F_v}})a_v$. On the other hand, two tuples $(\phi; \{a_v\}_{v \in \Xi})$ and $(\phi'; \{a'_v\}_{v \in \Xi})$ are equivalent if and only if there exists $b \in \mathfrak{g}'_\kappa$ such that

$$\phi' = \phi + (\mathbf{1} - \text{ad}^0(\bar{\rho}))b$$

and

$$a'_v = a_v + b$$

for every $v \in \Xi$. The assertion clearly follows.

For part (2), the proof of [CHT08, Corollary 2.2.13] goes through; cf. [Pat16, Proposition 9.2].

Part (3) follows from Lemma 3.1.1. $\square$

3.3. Fontaine–Laffaille deformations. In this subsection, we recall some results of the Fontaine–Laffaille lifting condition defined in [Boo19b]. Following [BK90, CHT08], we use a covariant version of the Fontaine–Laffaille theory [FL82].

Assume that ℓ is unramified in F , and let v be a finite place of F lying above ℓ . We denote by $\text{Fr} \in \text{Gal}(F_v/\mathbb{Q}_\ell)$ the absolute Frobenius element, and set $\Sigma_v := \text{Hom}_{\mathbb{Z}_\ell}(\mathcal{O}_v, \overline{\mathbb{Q}_\ell})$.

Assumption 3.3.1. The field E contains the images of all embeddings of F_v into $\overline{\mathbb{Q}_\ell}$.

Definition 3.3.2. Suppose Assumption 3.3.1 holds. We denote by $\mathcal{MF}_{\mathcal{O},v}$ the category of $\mathcal{O}_v \otimes_{\mathbb{Z}_\ell} \mathcal{O}$ -modules M of finite length equipped with

- a decreasing filtration $\{\text{Fil}^i M\}_{i \in \mathbb{Z}}$ by $\mathcal{O}_v \otimes_{\mathbb{Z}_\ell} \mathcal{O}$ -submodules that are $\mathcal{O}_v$ -direct summands, satisfying $\text{Fil}^0 M = M$ and $\text{Fil}^{\ell-1} M = 0$; and
- a Frobenius structure, that is, a $\text{Fr} \otimes 1$ -linear map $\Phi^i : \text{Fil}^i M \rightarrow M$ for every $i \in \mathbb{Z}$, satisfying the relations $\Phi^i|_{\text{Fil}^{i+1} M} = \ell \Phi^{i+1}$ and $\sum_{i \in \mathbb{Z}} \Phi^i \text{Fil}^i M = M$.

Definition 3.3.3. Suppose Assumption 3.3.1 holds. For each object M of $\mathcal{MF}_{\mathcal{O},v}$, the decomposition

$$\mathcal{O}_v \otimes_{\mathbb{Z}_\ell} \mathcal{O} = \bigoplus_{\tau \in \Sigma_v} \mathcal{O}$$

induces canonical decompositions

$$M = \bigoplus_{\tau \in \Sigma_v} M_\tau, \quad M_\tau := M \otimes_{\mathcal{O}_v \otimes_{\mathbb{Z}_\ell} \mathcal{O}, \tau \otimes 1} \mathcal{O}$$

and

$$\text{Fil}^i M = \bigoplus_{\tau \in \Sigma_v} \text{Fil}^i M_\tau, \quad \text{Fil}^i M_\tau = M_\tau \cap \text{Fil}^i M.$$

Then Φ^i induces $\mathcal{O}$ -linear maps

$$\Phi_\tau^i : \text{Fil}^i M_\tau \rightarrow M_{\tau \circ \text{Fr}^{-1}}.$$

for every $\tau \in \Sigma_v$ and every $i \in \mathbb{Z}$. We set

$$\text{gr}^i M_\tau := \text{Fil}^i M_\tau / \text{Fil}^{i+1} M_\tau,$$

$$\text{gr}^\bullet M_\tau := \bigoplus_{i \in \mathbb{Z}} \text{gr}^i M_\tau,$$

$$\text{gr}^\bullet M := \bigoplus_{\tau \in \Sigma_v} \text{gr}^\bullet M_\tau,$$

and define the set of τ -Fontaine–Laffaille weights of M to be

$$\text{HT}_\tau(M) := \{i \in \mathbb{Z} \mid \text{gr}^i M_\tau \neq 0\}.$$

We say that M has regular Fontaine–Laffaille weights if $\text{gr}^i M_\tau$ is generated over $\mathcal{O}$ by at most one element for every $\tau \in \Sigma_v$ and every $i \in \mathbb{Z}$.

Example 3.3.4. Suppose Assumption 3.3.1 holds. Each object R of $\mathcal{C}_O^f$ can be regarded as an object $R\{0\}$ of $\mathcal{MF}_{O,v}$ with

$$\mathrm{Fil}^i(R\{0\}) = \begin{cases} \mathcal{O}_v \otimes_{\mathbb{Z}_\ell} R & i \leq 0 \\ 0 & i > 0 \end{cases}.$$

We recall the notion of Tate twists on objects in $\mathcal{MF}_{O,v}$.

Definition 3.3.5. Suppose Assumption 3.3.1 holds. For each object M of $\mathcal{MF}_{O,v}$ and $s \in \mathbb{Z}$, we define $M\{s\}$ to be the objects of $\mathcal{MF}_{O,v}$ with $\mathrm{Fil}^i M\{s\} = \mathrm{Fil}^{i+s} M$ and $\Phi_{M\{s\}}^i = \Phi_M^{i+s}$.

For each integer b satisfying $0 \leq b \leq \ell - 2$, we let $\mathcal{MF}_{O,v}^{[0,b]}$ be the full subcategory of $\mathcal{MF}_{O,v}$ consisting of objects M satisfying $\mathrm{Fil}^{b+1} M = 0$. Moreover, for each object R of $\mathcal{C}_O^f$, we let $\mathcal{MF}_{O,v}^{[0,b]}|_R$ be the full subcategory of $\mathcal{MF}_{O,v}^{[0,b]}$ consisting of objects M such that M is of finite free over $\mathcal{O}_v \otimes_{\mathbb{Z}_\ell} R$ and that $\mathrm{Fil}^i M$ is a direct summand of M as a $\mathcal{O}_v \otimes_{\mathbb{Z}_\ell} R$ -submodule for every $i \in \mathbb{Z}$.

Proposition 3.3.6. Suppose Assumption 3.3.1 holds. Let $\mathcal{O}[\mathrm{Gal}_{F_v}]^{\mathrm{f.l.}}$ be the category of $\mathcal{O}$ -modules of finite length equipped with a continuous action of Gal_{F_v} . There exists an exact fully faithful covariant $\mathcal{O}$ -linear functor

$$\mathbf{T}_v^{\mathrm{cris}} : \mathcal{MF}_{O,v} \rightarrow \mathcal{O}[\mathrm{Gal}_{F_v}]^{\mathrm{f.l.}}$$

satisfying that

- (1) The essential image of $\mathbf{T}_v^{\mathrm{cris}}$ is closed under taking subobjects and quotient objects.
- (2) For each object M of $\mathcal{MF}_{O,v}$ and each $s \in \mathbb{Z}$, $\mathbf{T}_v^{\mathrm{cris}}(M\{s\})$ is isomorphic to $\mathbf{T}_v^{\mathrm{cris}}(M)(-s)$.
- (3) For each object M of $\mathcal{MF}_{O,v}$ of finite length (as $\mathcal{O}$ -module), we have

$$\mathrm{length}_{\mathcal{O}} M = [F_v : \mathbb{Q}_\ell] \cdot \mathrm{length}_{\mathcal{O}} \mathbf{T}_v^{\mathrm{cris}}(M).$$

- (4) For each object R of $\mathcal{C}_O^f$ and each object M of $\mathcal{MF}_{O,v}|_R$, $\mathbf{T}_v^{\mathrm{cris}}(M)$ is finite free over R .

Proof. Part (1) follows from [Boo19b, Fact 4.10]. Part (2) follows from the definition; see [Boo19b, Definition 4.9]. Part (3) follows from [Boo19b, Lemma 4.14]. Part (4) follows from [Boo19b, Lemma 4.15]. $\square$

Definition 3.3.7. Let $a \leq b$ be integers such that $b - a \leq \ell - 2$. For an object R of $\mathcal{C}_O$ and a continuous representation $\rho : \mathrm{Gal}_{F_v} \rightarrow G(R)$, we say that ρ is *crystalline with regular Fontaine–Laffaille weights in $[a, b]$* if there exists a finite unramified extension $E_\dagger$ of E that splits the extension $F_v/\mathbb{Q}_\ell$, with ring of integers $\mathcal{O}_\dagger$, such that for every quotient R' of R in $\mathcal{C}_O^f$, $((\mathrm{Std} \circ \rho)(a) \otimes_{\mathcal{O}} \mathcal{O}_\dagger) \otimes_R R'$ is isomorphic to $\mathbf{T}_v^{\mathrm{cris}}(M')$ for some object M' of $\mathcal{MF}_{O_\dagger, v}^{[0, b-a]}|_{R'}$ with regular Fontaine–Laffaille weights.

Definition 3.3.8. Suppose $(\Gamma, \bar{\rho}, \chi)$ is a triple as in Setup 3.2.1 in which $\Gamma = \mathrm{Gal}_{F_v}$ and $\chi = \varepsilon_\ell^{2-N}$, such that $\mathrm{Std} \circ \bar{\rho}$ is crystalline with regular Fontaine–Laffaille weights in $[a, b]$ for some integers $a \leq b$ satisfying $b - a \leq \frac{\ell-2}{2}$. We define $\mathcal{D}^{\mathrm{FL}, \square}$ to be the local lifting condition (with respect to the pair $(\bar{\rho}, \chi)$) that classifies regular Fontaine–Laffaille crystallines liftings of $\bar{\rho}$ (see Definition 3.3.7). It is a local lifting condition by [Boo19b, Corollary 5.6].

Proposition 3.3.9. Let $(\Gamma, \bar{\rho}, \chi)$ be a triple as in Definition 3.3.8. If B_κ is a Borel subgroup of G_κ , then the local lifting condition $\mathcal{D}^{\mathrm{FL}, \square}$ is formally smooth over $\mathrm{Spf} \mathcal{O}$ of pure relative dimension

$$\dim G'_\kappa + [F_v : \mathbb{Q}_\ell] \cdot (\dim G_\kappa - \dim B_\kappa).$$

In particular, it follows from Equation 3.1 that

$$\dim_\kappa L(\mathcal{D}^{\mathrm{FL}}) = \dim_\kappa H^0(F_v, \mathrm{ad}^0(\bar{\rho})) + [F_v : \mathbb{Q}_\ell](\dim G_\kappa - \dim B_\kappa).$$

Proof. By descent for formal smoothness (see [Sta18, 06CM]), we may assume that Assumption 3.3.1 holds. Then the assertion follows from [Boo19b, Theorem 5.2]. $\square$

3.4. Tame representations. In this subsection, we recall some results on minimally ramified deformations of tamely ramified self-dual local Galois representations.

We fix a positive integer q that is coprime to ℓ , and note that all results in this subsection adapt verbatim to the case where G is replaced by the smooth group scheme GO_{2m+1} defined by the standard symmetric pairing given by the matrix J_{2m+1} , cf. [Boo19a, §5.2].

Definition 3.4.1. The q -tame group, denoted by T_q , is defined to be the semidirect product topological group $t^{\mathbb{Z}_\ell} \rtimes \phi_q^{\widehat{\mathbb{Z}}}$, satisfying $\phi_q t \phi_q^{-1} = t^q$.

We recall the following result on the universal deformations of the tame group T_q from [LTX⁺24, Proposition 3.16].

Proposition 3.4.2. Suppose $(\Gamma, \bar{\rho}, \chi)$ is a triple as in Setup 3.2.1 in which $\Gamma = T_q$. Then the universal lifting ring $R_{\bar{\rho}}^{\square}$ (see Definition 3.2.2) is a local complete intersection, flat and of pure relative dimension $\dim G'_\kappa$ over $\mathcal{O}$.

For the rest of this subsection, we assume $\ell > 2m$. Denote by $\mathcal{N}_{2m}$ (resp. $\mathcal{U}_{2m}$) the closed subscheme of Mat_{2m} (resp. GL_{2m}) defined by the equation $X^{2m} = 0$ (resp. $(A - 1)^{2m} = 0$). For every object R of $\mathcal{C}_{\mathcal{O}}$, there is a truncated exponential map

$$\exp : \mathcal{N}_{2m}(R) \rightarrow \mathcal{U}_{2m}(R) : X \mapsto 1 + X + \dots + \frac{X^{2m-1}}{(2m-1)!},$$

which is a bijection. Its inverse is given by the truncated logarithmic map

$$\log : \mathcal{U}_{2m}(R) \rightarrow \mathcal{N}_{2m}(R) : A \mapsto - \sum_{i=1}^{2m-1} \frac{(1-A)^i}{i}.$$

Furthermore, for any $C \in \mathrm{GL}_{2m}(R)$ and any positive integer k , we have

$$\exp(CAC^{-1}) = C \exp(A) C^{-1}, \quad \log(CAC^{-1}) = C \log(A) C^{-1}$$

and

$$\exp(kA) = \exp(A)^k, \quad \log(B^k) = k \log(B);$$

cf. [Tay08, Lemma 2.4].

For the rest of this subsection, we suppose $(\Gamma, \bar{\rho}, \chi)$ is a triple as in Setup 3.2.1 in which $\Gamma = T_q$. Then $\bar{A} := \bar{\rho}(t)$ is unipotent, as it has ℓ -power torsion in the image of $\bar{\rho}$ by the continuity of $\bar{\rho}$. Set $\bar{X} := \log A \in \mathcal{N}_{2m}(\kappa)$.

Definition 3.4.3. Let $\sigma \vdash 2m$ be a partition $2m = m_1 + \dots + m_r$ of $2m$. It is called *admissible* if those m_i satisfying $m_i \equiv N \pmod{2}$ appears for an even number of times. Let $\mathfrak{P}_{2m}$ denote the set of admissible partitions of $2m$.

By the classification of nilpotent orbits of G , for each $L \in \{\bar{\kappa}, \overline{\mathbb{Q}_\ell}\}$, there exists canonical surjective maps $\pi : \mathcal{N}_{2m}(L) \rightarrow \mathfrak{P}_{2m}$ such that the fibers of π are exactly the orbits in $\mathcal{N}_{2m}(L)$ under the conjugate action of $G(L)$; see [Jan04, Theorem 1.6]. The corresponding orbit is the intersection of $\mathfrak{g}_L \subset \mathfrak{gl}_{2m,L}$ with the $\mathrm{GL}_{2m,L}$ -orbit corresponding to that partition of $2m$.

We introduce the following convenient assumption.

Assumption 3.4.4. κ contains a subfield κ^b of degree two such that $\mathrm{Std} \circ \bar{\rho}$ is defined over κ^b .

By [Boo19a, §3.1], when Assumption 3.4.4 holds, we obtain an element $X_0 \in \mathcal{N}_{2m}(\mathcal{O})$ such that

- $\pi(X_0) = \pi(\bar{X})$, where we regard X_0 as an element in $\mathcal{N}_{2m}(\overline{\mathbb{Q}_\ell})$;
- the centralizer $Z_G(X_0) \subset G$ is smooth over $\mathcal{O}$; and
- there exists an element $\Phi \in G(\mathcal{O})$ such that $\mathrm{Ad}_G(\Phi)X_0 = qX_0$.

Following [Boo19a, §3.2], we obtain a functor

$$\mathrm{Nil}_{\bar{X}} : \mathcal{C}_{\mathcal{O}} \rightarrow \mathbf{Set} : R \mapsto \{X \in \mathfrak{g}_R : \mathrm{Ad}_G(g)(X_0) = X \text{ for some } g \in G^\wedge(\mathcal{O})\},$$

which is representable. It follows from [Boo19a, Corollary 5.2] that we obtain an exponential map

$$\exp : \mathrm{Nil}_{\bar{X}} \rightarrow G$$

such that for any object R of $\mathcal{C}_O$ and $g \in G^\wedge(R)$, $X \in \text{Nil}_{\overline{X}}(R)$ and any positive integer k , we have

$$\exp(kX) = \exp(X)^k, \quad \text{Ad}(g) \exp(X) = \exp(\text{Ad}_G(g)X).$$

Definition 3.4.5. Suppose Assumption 3.4.4 holds. We say that a lifting ρ of $\bar{\rho}$ to an object R of $\mathcal{C}_O$ is *minimally ramified* if there exists an element $X \in \text{Nil}_{\overline{X}}(R)$ such that $\rho(t) = \exp(X)$.

Let $\mathcal{D}^{\min, \square}$ be the local lifting condition of $\bar{\rho}$ with respect to χ (see Definition 3.2.3) that classifies minimally ramified liftings of $\bar{\rho}$.

We recall the following result of Booher [Boo19a] concerning minimally ramified deformation problems.

Proposition 3.4.6. *Suppose that $\ell > 2m$ and Assumption 3.4.4 holds. Then the local lifting condition $\mathcal{D}^{\min, \square}$ is formally smooth over $\text{Spf } \mathcal{O}$ of pure relative dimension $\dim G'_\kappa$. In particular, it follows from Equation (3.1) that*

$$\dim_\kappa L(\mathcal{D}^{\min}) = \dim_\kappa H^0(T_q, \text{ad}^0(\bar{\rho})).$$

Proof. This follows from the proofs of [Boo19a, Proposition 5.6, Corollary 5.8]. $\square$

We recall the following lemma on the decomposition of liftings of representations of tame groups that will be useful later.

Lemma 3.4.7. *Let $(\bar{r}, \bar{M})$ be an unramified representation of the tame group T_q over κ . Suppose that $\bar{M}$ admits a decomposition*

$$(3.3) \quad \bar{M} = \bar{M}_1 \oplus \dots \oplus \bar{M}_s$$

stable under the action of $\bar{r}(\phi_q)$ such that the characteristic polynomials of $\bar{r}(\phi_q)$ on $\bar{M}_i$ are pairwise coprime for $1 \leq i \leq s$. Let (r, M) be a lifting of $(\bar{r}, \bar{M})$ to an object R of $\mathcal{C}_O$.

(1) *There is a unique decomposition*

$$M = M_1 \oplus \dots \oplus M_s$$

of free R -modules lifting (3.3), such that each M_i is stable under the action of $\bar{r}(\phi_q)$.

(2) *Suppose q is not an eigenvalue for the canonical action of ϕ_q on $\text{Hom}_\kappa(\bar{M}_i, \bar{M}_j)$ for all $i \neq j$, then the decomposition in (1) is stable under the whole group T_q .*

Proof. For (1): Let $f(T)$ be the characteristic polynomial of $\rho(\phi_q)$ on M . Its reduction decomposes as $\bar{f}(T) = \prod_{i=1}^r g_i(T)$, where $g_i(T)$ is the characteristic polynomial of $\bar{\rho}(\phi_q)$ on M_i for every $1 \leq i \leq r$. By hypothesis, these characteristic polynomials $g_i(T)$ are pairwise coprime. So it follows from Hensel's lemma that there exists a decomposition $f(T) = \prod_{i=1}^r \tilde{g}_i(T)$ in which $\tilde{g}_i(T) \in R[T]$ are polynomials lifting $g_i(T)$ for every $1 \leq i \leq r$. For each $1 \leq i \leq r$, we define $\tilde{Q}_i = \prod_{i \neq j} \tilde{P}_i$, and choose $A_i, B_i \in R[T]$ with $A_i \tilde{P}_i + B_i \tilde{Q}_i = 1$. Then $e_i := B(\rho(\phi_q)) \tilde{Q}_i(\rho(\phi_q))$ are projectors on M satisfying $\sum_i e_i = 1$. Then we finish by taking $M_i := e_i M$.
Part (2) is proved in [LT⁺24, Lemma 3.21]. $\square$

3.5. Minimally ramified representations. In this subsection, we recall the notion of minimally ramified deformations.

Let v be a finite place of F that is not in $\Sigma(\ell)$. Let P_v be the maximal subgroup of I_{F_v} of pro-order coprime to ℓ , and set $T_v := \text{Gal}_{F_v} / P_v$, which is isomorphic to the $\|v\|$ -tame group $T_{\|v\|}$ (see Definition 3.4.1). Suppose $(\Gamma, \bar{\rho}, \chi)$ is a triple as in Setup 3.2.1 in which $\Gamma = \text{Gal}_{F_v}$.

For any irreducible representation τ of P_v with coefficients in κ , we define

$$\Gamma_\tau := \{\sigma \in \text{Gal}_{F_v} \mid \tau^\sigma \cong \tau\},$$

which is a subgroup of Gal_{F_v} containing P_v . Let T_τ be the image of Γ_τ in T_v .

Lemma 3.5.1. *Suppose τ is an irreducible representation of P_v with coefficients in κ .*

- (1) *The dimension of τ is coprime to ℓ ; and τ has a unique deformation to a representation $\tilde{\tau}$ of P_v over $\mathcal{O}$;*
- (2) *The deformation $\tilde{\tau}$ in (1) admits a unique extension to $\Gamma_\tau \cap I_{F_v}$ over $\mathcal{O}$ with tame determinant (that is, the determinant maps the tame generator to a root of unity with finite order coprime to ℓ);*
- (3) *There exists an extension of the deformation $\tilde{\tau}$ in (2) to a representation of Γ_τ over $\mathcal{O}$.*

Proof. This is [CHT08, Lemma 2.4.11]. $\square$

For each irreducible representation τ of P_v with coefficients in κ and each lifting ρ of $\bar{\rho}$ to an object R of $\mathcal{C}_{\mathcal{O}}$, we set

$$W_{\tau}(\rho) := \text{Hom}_{R[P_v]}(\tilde{\tau} \otimes_{\mathcal{O}} R, \text{Std} \circ \rho),$$

where $\tilde{\tau}$ is a representation of Γ_{τ} with coefficients in $\mathcal{O}$ from Lemma 3.5.1(3) (later we will fix such choices in Proposition 3.5.4). Note that $W_{\tau}(\rho)$ is a finite free R -module equipped with the induced continuous action by Γ_{τ} , and $\tilde{\tau} \otimes_{\mathcal{O}} W_{\tau}(\rho)$ is equipped with a natural action by Γ_{τ} .

Assumption 3.5.2. κ contains a subfield κ^b of degree 2 such that every irreducible summand of $(\text{Std} \circ \bar{\rho})|_{P_v} \otimes_{\kappa} \bar{\kappa}$ is defined over κ^b .

Definition 3.5.3. Assume Assumption 3.5.2 holds. We denote by $\mathfrak{T}$ the set of isomorphism classes of irreducible representations τ of P_v such that $W_{\tau}(\bar{\rho}) \neq 0$. Then Gal_{F_v} acts on $\mathfrak{T}$ by conjugation, whose orbits we denote by $\mathfrak{T}/\text{Gal}_{F_v}$. For each $\tau \in \mathfrak{T}$, we denote by $[\tau]$ its orbit in $\mathfrak{T}/\text{Gal}_{F_v}$.

We denote the underlying space of $\text{Std} \circ \bar{\rho}$ by V , and define $V^* := V^{\vee} \otimes \chi$. Then ρ defines an isomorphism of $\kappa[\text{Gal}_{F_v}]$ -modules $\psi : V \cong V^*$. By comparing P_v -isotypic components, we get a natural isomorphism of $\kappa[\text{Gal}_{F_v}]$ -modules

$$\psi_{\tau} : W_{\tau}(\bar{\rho})^* \cong W_{\tau^*}(\bar{\rho})$$

for some $\tau^* \in \mathfrak{T}$.

Then we construct a Gal_{F_v} -stable partition $\mathfrak{T} = \mathfrak{T}_1 \coprod \mathfrak{T}_2 \coprod \mathfrak{T}_3$ as follows.

- (1) We define $\mathfrak{T}_1$ to be the subset of $\mathfrak{T}$ consisting of those τ such that $[\tau] \neq [\tau^*]$. We also fix a subset $\mathfrak{T}_1^{\heartsuit} \subset \mathfrak{T}_1$ such that $\{\tau, \tau^* | \tau \in \mathfrak{T}_1^{\heartsuit}\}$ is set of representatives for the Gal_{F_v} -action on $\mathfrak{T}_1$.
- (2) We define $\mathfrak{T}_2$ to be the subset of $\mathfrak{T}$ consisting of those τ such that $\tau \cong \tau^*$ as P_v -representations. We also fix a subset $\mathfrak{T}_2^{\heartsuit} \subset \mathfrak{T}_2$ of representatives for the Gal_{F_v} -action on $\mathfrak{T}_2$.
- (3) We define $\mathfrak{T}_3$ to be the subset of $\mathfrak{T}$ consisting of those τ such that $[\tau] = [\tau^*]$ but $\tau \not\cong \tau^*$ as P_v -representations. We also fix a subset $\mathfrak{T}_3^{\heartsuit} \subset \mathfrak{T}_3$ of representatives for the Gal_{F_v} -action on $\mathfrak{T}_3$.

Proposition 3.5.4. Assume Assumption 3.5.2 holds.

- (1) For each $\tau \in \mathfrak{T}_1^{\heartsuit}$, we fix a representation $\tilde{\tau}$ of Γ_{τ} with coefficients in $\mathcal{O}$ from Lemma 3.5.1(3). We set

$$\overline{G}_{\tau} := \underline{\text{Aut}}(W_{\tau}(\bar{\rho})).$$

- (2) For each $\tau \in \mathfrak{T}_2^{\heartsuit}$, any isomorphism $\iota_{\tau} : \tau \xrightarrow{\sim} \tau^*$ induces a sign-symmetric pairing (for some fixed sign ϵ_{τ}) on τ , cf. [Boo19a, Lemma 6.5]. Moreover, there is a sign-symmetric perfect pairing $\langle -, - \rangle_{W(\bar{\rho})}$ on $W(\bar{\rho})$, and we set

$$\overline{G}_{\tau} := \underline{\text{GAut}}(W_{\tau}(\bar{\rho}), \langle -, - \rangle_{W_{\tau}(\bar{\rho})}),$$

which is a split general symplectic group or split general orthogonal group over κ . It follows from [Boo19a, Lemma 6.10, Lemma 6.11] that we can fix a lifting $\tilde{\tau}$ of τ as in Lemma 3.5.1(3) with an isomorphism $\tilde{\tau} \xrightarrow{\sim} \tilde{\tau}^*$ lifting that on ι_{τ} .

- (3) For each $\tau \in \mathfrak{T}_3^{\heartsuit}$, we define

$$\Gamma_{\tau \oplus \tau^*} := \{g \in \text{Gal}_{F_v} : (\tau \oplus \tau^*)^g \cong \tau \oplus \tau^*\},$$

which contains Γ_{τ} as a subgroup of index 2, as conjugation either preserves τ and τ^* or swaps them. Any isomorphism $\iota_{\tau} : (\tau \oplus \tau^*) \xrightarrow{\sim} (\tau \oplus \tau^*)^*$ of $\kappa[\Gamma_{\tau \oplus \tau^*}]$ -modules induces a sign-symmetric pairing on $W_{\tau}(\bar{\rho})$, and we set

$$\overline{G}_{\tau} := \underline{\text{GAut}}(W_{\tau}(\bar{\rho}), \langle -, - \rangle_{W_{\tau}(\bar{\rho})}),$$

which is a split general symplectic group or split general orthogonal group over κ . Then we obtain a perfect pairing on $\tau \oplus \tau^*$, cf. [Boo19a, p. 30]. It follows from [Boo19a, Lemma 6.12] that we can fix a lifting $\widetilde{\tau \oplus \tau^*}$ of $\tau \oplus \tau^*$ with an isomorphism $\widetilde{\tau \oplus \tau^*} \xrightarrow{\sim} (\widetilde{\tau \oplus \tau^*})^*$ lifting that on ι_{τ} .

In each case, we can lift $\overline{G}_{\tau}$ to a split reductive group G_{τ} over $\mathcal{O}$ by lifting the linear algebra data.

Proof. These follow from [Boo19a, §6.3]. $\square$

Proposition 3.5.5. Assume Assumption 3.5.2 holds. Suppose R is an object of $\mathcal{C}_O$ and ρ is a lifting of $\bar{\rho}$ to R .

(1) Recall that we have defined

$$W_\tau(\rho) := \text{Hom}_{R[P_v]}(\tilde{\tau} \otimes_O R, \text{Std} \circ \rho)$$

for each $\tau \in \mathfrak{T}_1^\heartsuit$, which is a finite free R -module of some rank $m_\tau \geq 1$ equipped with the induced continuous action by T_τ .

(2) Recall that we have defined

$$W_\tau(\rho) := \text{Hom}_{R[P_v]}(\tilde{\tau} \otimes_O R, \text{Std} \circ \rho)$$

for each $\tau \in \mathfrak{T}_2^\heartsuit$, which is a finite free R -module of some rank $m_\tau \geq 1$ equipped with the induced continuous action by T_τ .

(3) We define

$$W_{\tau \oplus \tau^*}(\rho) := \text{Hom}_{R[P_v]}(\widetilde{\tau \oplus \tau^*} \otimes_O R, \text{Std} \circ \rho),$$

which is a finite free R -module of some rank $m_\tau \geq 1$ equipped with the induced continuous action by $T_{\tau \oplus \tau^*}$.

Then we have a decomposition of $R[\text{Gal}_{F_v}]$ -modules

$$\begin{aligned} \text{Std} \circ \rho = & \bigoplus_{\tau \in \mathfrak{T}_1^\heartsuit} \left(\text{Ind}_{\Gamma_\tau}^{\text{Gal}_{F_v}}(\tilde{\tau} \otimes_O W_\tau(\rho)) \oplus \text{Ind}_{\Gamma_{\tau^*}}^{\text{Gal}_{F_v}}(\tilde{\tau}^* \otimes_O W_{\tau^*}(\rho)) \right) \\ & \oplus \bigoplus_{\tau \in \mathfrak{T}_2^\heartsuit} \text{Ind}_{\Gamma_\tau}^{\text{Gal}_{F_v}}(\tilde{\tau} \otimes_O W_\tau(\rho)) \oplus \bigoplus_{\tau \in \mathfrak{T}_3^\heartsuit} \text{Ind}_{\Gamma_{\tau \oplus \tau^*}}^{\text{Gal}_{F_v}}(\widetilde{\tau \oplus \tau^*} \otimes_O W_{\tau \oplus \tau^*}(\rho)). \end{aligned}$$

Moreover, for $\tau \in \mathfrak{T}_2^\heartsuit$ (resp. $\mathfrak{T}_3^\heartsuit$), we have natural sign-symmetric pairing on $W_\tau(\rho)$ (resp. $W_{\tau \oplus \tau^*}(\rho)$).

Proof. These follow from [Boo19a, §6.3]. $\square$

Proposition 3.5.6. Assume Assumption 3.5.2 holds. We also keep the choices of Definition 3.5.3 and the isomorphisms $\iota_\tau : \tau \xrightarrow{\sim} \tau^*$ as in Proposition 3.5.4(2). Then for every object R of $\mathcal{C}_O$, the assignment

$$\rho \mapsto \left((W_\tau(\rho))_{\tau \in \mathfrak{T}_1^\heartsuit}, (W_\tau(\rho))_{\tau \in \mathfrak{T}_2^\heartsuit}, (W_{\tau \oplus \tau^*}(\rho))_{\tau \in \mathfrak{T}_3^\heartsuit} \right)$$

induces a bijection between deformations of $\bar{\rho}$ to R (with respect to χ) and equivalence classes of tuples $\left((W_\tau)_{\tau \in \mathfrak{T}_1^\heartsuit}, (W_\tau)_{\tau \in \mathfrak{T}_2^\heartsuit}, (W_{\tau \oplus \tau^*})_{\tau \in \mathfrak{T}_3^\heartsuit} \right)$ where

- (1) for $\tau \in \mathfrak{T}_1^\heartsuit$, $W_\tau : T_\tau \rightarrow \text{GL}_{m_\tau}(R)$ is a continuous homomorphism that reduces to $W_\tau(\bar{\rho})$;
- (2) for $\tau \in \mathfrak{T}_2^\heartsuit$, $W_\tau : T_\tau \rightarrow G_\tau(R)$ is a continuous homomorphism that reduces to $W_\tau(\bar{\rho})$ with trivial similitude character; and
- (3) for $\tau \in \mathfrak{T}_3^\heartsuit$, $W_{\tau \oplus \tau^*} : T_{\tau \oplus \tau^*} \rightarrow G_\tau(R)$ is a continuous homomorphism that reduces to $W_{\tau \oplus \tau^*}(\bar{\rho})$ with trivial similitude character.

Here, two tuples

$$\left((W_\tau)_{\tau \in \mathfrak{T}_1^\heartsuit}, (W_\tau)_{\tau \in \mathfrak{T}_2^\heartsuit}, (W_{\tau \oplus \tau^*})_{\tau \in \mathfrak{T}_3^\heartsuit} \right), \left((W'_\tau)_{\tau \in \mathfrak{T}_1^\heartsuit}, (W'_\tau)_{\tau \in \mathfrak{T}_2^\heartsuit}, (W'_{\tau \oplus \tau^*})_{\tau \in \mathfrak{T}_3^\heartsuit} \right)$$

are said to be equivalent if

- (1) for $\tau \in \mathfrak{T}_1^\heartsuit$, W_τ and W'_τ are conjugate by an element of $\text{GL}_{m_\tau}^\wedge(R)$;
- (2) for $\tau \in \mathfrak{T}_2^\heartsuit$, W_τ and W'_τ are conjugate by an element of $(G'_\tau)^\wedge(R)$, where G'_τ is the derived subgroup of G_τ ; and
- (3) for $\tau \in \mathfrak{T}_3^\heartsuit$, $W_{\tau \oplus \tau^*}$ and $W'_{\tau \oplus \tau^*}$ are conjugate by an element of $(G'_\tau)^\wedge(R)$, where G'_τ is the derived subgroup of G_τ .

Proof. The proof of [Boo19a, Proposition 6.15] goes through, cf. [LTX⁺24, Proposition 3.28]. $\square$

We record the following corollary that will be useful later.

Corollary 3.5.7. *Suppose $\ell \nmid \prod_{i=1}^{2m} (\|v\|^i - 1)$. Let $\iota_\ell : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}_\ell}$ be an isomorphism, and R an object of $\mathcal{C}_O$ contained in $\overline{\mathbb{Q}_\ell}$. Let ρ_1, ρ_2 be two liftings of ρ to R , and let Π_1, Π_2 be two irreducible admissible representations of $\mathrm{GL}_{2m}(F_v)$ such that $\mathrm{WD}(\mathrm{Std} \circ \rho_i \otimes_R \overline{\mathbb{Q}_\ell})^{F\text{-ss}} = \mathrm{rec}_{2m}(\iota_\ell \Pi_i)$ for every $i \in \{1, 2\}$. Then Π_1 and Π_2 are in the same Bernstein component.*

Proof. Using Proposition 3.5.5, the proof of [LTX⁺24, Corollary 3.31] goes through. $\square$

We can now drop the Assumption 3.5.2 and define minimally ramified liftings of $\bar{\rho}$ for general coefficient field E .

Definition 3.5.8. Suppose $\ell > 2m$. Choose a finite unramified extension $E_\dagger$ of E contained in $\overline{\mathbb{Q}_\ell}$ contained in $\overline{\mathbb{Q}_\ell}$ that satisfies Assumption 3.4.4 with ring of integers $\mathcal{O}_\dagger$ and residue field $\kappa_\dagger$. We also keep the choices of Definition 3.5.3 and the isomorphisms $\iota_\tau : \tau \xrightarrow{\sim} \tau^*$ as in Proposition 3.5.4(2). Let ρ be a lifting of $\bar{\rho}$ to an object R of $\mathcal{C}_O$, then we obtain the decomposition of $R[\mathrm{Gal}_F]$ -modules

$$\begin{aligned} \mathrm{Std} \circ \rho = & \bigoplus_{\tau \in \mathfrak{T}_1^\heartsuit} \left(\mathrm{Ind}_{\Gamma_\tau}^{\mathrm{Gal}_{F_v}} (\tilde{\tau} \otimes_{\mathcal{O}} W_\tau(\rho)) \oplus \mathrm{Ind}_{\Gamma_{\tau^*}}^{\mathrm{Gal}_{F_v}} (\tilde{\tau}^* \otimes_{\mathcal{O}} W_{\tau^*}(\rho)) \right) \\ & \oplus \bigoplus_{\tau \in \mathfrak{T}_2^\heartsuit} \mathrm{Ind}_{\Gamma_\tau}^{\mathrm{Gal}_{F_v}} (\tilde{\tau} \otimes_{\mathcal{O}} W_\tau(\rho)) \oplus \bigoplus_{\tau \in \mathfrak{T}_3^\heartsuit} \mathrm{Ind}_{\Gamma_{\tau \oplus \tau^*}}^{\mathrm{Gal}_{F_v}} (\widetilde{\tau \oplus \tau^*} \otimes_{\mathcal{O}} W_{\tau \oplus \tau^*}(\rho)). \end{aligned}$$

We say that ρ is minimally ramified with similitude χ if the following conditions hold:

- (1) For each $\tau \in \mathfrak{T}_1^\heartsuit$, $W_\tau(\rho)$ is a minimally ramified extension of $W_\tau(\bar{\rho})$ in the sense of [CHT08, Definition 2.4.14].
- (2) For each $\tau \in \mathfrak{T}_2^\heartsuit$ (resp. $\tau \in \mathfrak{T}_3^\heartsuit$), W_τ (resp. $W_{\tau \oplus \tau^*}$) is minimally ramified in the sense of Definition 3.4.5 as a representation of T_τ (resp. $T_{\tau \oplus \tau^*}$) valued in the group G_τ with trivial similitude character.

Remark 3.5.9. It is easy to check that Definition 3.5.8 doesn't depend on the choices of $E_\dagger$ and the choices made in Definition 3.5.3 and Proposition 3.5.4.

Definition 3.5.10. When $\ell > 2m$, let $\mathcal{D}^{\min, \square}$ denote the local lifting condition of $\bar{\rho}$ (with respect to χ) that classifies minimally ramified liftings of ρ (see Definition 3.5.8).

Proposition 3.5.11. *The following statements hold:*

- (1) *The universal lifting ring $R_\rho^\square \in \mathcal{C}_O$ is a reduced local complete intersection, flat and of pure relative dimension $\dim G'_\kappa$ over $\mathcal{O}$.*
- (2) *Every irreducible component of $\mathrm{Spf} R_\rho^\square$ is a local lifting condition (see Definition 3.2.2).*
- (3) *When $\ell > 2m$, the local lifting condition $\mathcal{D}^{\min, \square}$ is an irreducible component of $R_\rho^\square$ that is formally smooth over $\mathrm{Spf} \mathcal{O}$ of pure relative dimension $\dim G'_\kappa$. In particular, it follows from Equation (3.1) that*

$$\dim_\kappa L(\mathcal{D}^{\min}) = \dim_\kappa H^0(\mathrm{Gal}_{F_v}, \mathrm{ad}^0(\bar{\rho})).$$

Proof. For this proposition, we may assume that E satisfies Assumption 3.5.2.

For (1), we define for each $\tau \in \mathfrak{T}_1^\heartsuit \amalg \mathfrak{T}_2^\heartsuit \amalg \mathfrak{T}_3^\heartsuit$ the universal lifting ring $R_\tau^\square$ representing the lifting condition of $W_\tau(\bar{\rho})$ or $W_{\tau \oplus \tau^*}(\bar{\rho})$ valued in G_τ (with trivial similitude character when $\tau \in \mathfrak{T}_2^\heartsuit \amalg \mathfrak{T}_3^\heartsuit$), see Definition 3.2.2. It follows from Proposition 3.5.6 that $R_\rho^\square$ is a power series ring over

$$\widehat{\bigotimes_{\tau \in \mathfrak{T}_1^\heartsuit \amalg \mathfrak{T}_2^\heartsuit \amalg \mathfrak{T}_3^\heartsuit} R_\tau^\square}.$$

For each $\tau \in \mathfrak{T}_1^\heartsuit \amalg \mathfrak{T}_2^\heartsuit \amalg \mathfrak{T}_3^\heartsuit$, we know $R_\tau^\square$ is a local complete intersection, flat and of pure relative dimension $\dim G'_\kappa$ over $\mathrm{Spf} \mathcal{O}$ by [Sho18, Theorem 2.5] for $\tau \in \mathfrak{T}_1^\heartsuit$, and by Proposition 3.4.2 for $\tau \in \mathfrak{T}_2^\heartsuit \amalg \mathfrak{T}_3^\heartsuit$. On the other hand, $R_\rho^\square[\frac{1}{\ell}]$ is a local complete intersection, generically regular and of pure dimension $\dim G'_\kappa$ by [BG19, Theorem 3.3.2] or [BP19, Theorem 1]. Thus $R_\rho^\square$ is also a local complete intersection, flat and equidimensional of dimension $\dim G'_\kappa$ over $\mathcal{O}$. Since $R_\rho^\square$ is generically reduced and Cohen–Macaulay, it is reduced. (1) is proved.

For (2): take an irreducible component $\mathcal{D}^\square$ of $\mathrm{Spf} \mathcal{R}_{\bar{\rho}}^\square$, then the conjugation action induces a homomorphism $f : (G')^\wedge \times_{\mathrm{Spf} \mathcal{O}} \mathcal{D}^\square \rightarrow \mathrm{Spf} \mathcal{R}_{\bar{\rho}}^\square$, whose image contains $\mathcal{D}^\square$. Since the source of f is irreducible, and $\mathrm{Spf} \mathcal{R}_{\bar{\rho}}^\square$ is reduced, the image of f has to be $\mathcal{D}^\square$. In other words, $\mathcal{D}^\square$ is a local lifting condition.

(3): Since $\mathcal{D}^{\min, \square}$ is Zariski closed in $\mathrm{Spf} \mathcal{R}_{\bar{\rho}}^\square$, it suffices to show that $\mathcal{D}^{\min, \square}$ is formally smooth over $\mathrm{Spf} \mathcal{O}$ of pure relative dimension $\dim G'_\kappa$. This claim follows from [Boo19a, Corollary 6.16]. $\square$

3.6. Level-raising deformations. In this subsection, we study level-raising deformations. Assume that $N = 2m + 1$ is odd and that $\ell > 2m$. We work in the following setting.

Setup 3.6.1.

- Let v be a finite place of F such that $\ell \nmid (\|v\|^2 - 1) \|v\|$.
- Let P_v be the maximal subgroup of I_{F_v} of pro-order coprime to ℓ , and set $T_v := \mathrm{Gal}_{F_v} / P_v$, which is isomorphic to the $\|v\|$ -tame group $T_{\|v\|}$ (see Definition 3.4.1).
- Let $(\Gamma, \bar{\rho}, \chi)$ be a triple as in Setup 3.2.1, satisfying that $\Gamma = \mathrm{Gal}_{F_v}$, $\chi = \varepsilon_\ell^{1-2m}$, $\bar{\rho}$ is unramified, and that

$$\dim_\kappa \ker (\mathrm{Std} \circ \bar{\rho}_v(\phi_v) - \|v\|^{-m})^{2m} = 1.$$

By Lemma 3.5.1(1), every lifting ρ of $\bar{\rho}$ to an object R of $\mathcal{C}_{\mathcal{O}}$ (with respect to χ) factors through $T_{\|v\|}$. Moreover, it follows from Lemma 3.4.7 that for such a lifting ρ , if we fix an isomorphism between the underlying space of $\mathrm{Std} \circ \rho$ with R^{2m} , then we obtain a canonical decomposition

$$(3.4) \quad R^{\oplus(2m)} = M_0 \oplus M_1$$

of free R -modules stable under the action of $\rho(\phi_v)$, such that

$$\det(T - \mathrm{Std} \circ \rho(\phi_v)|M_0 \otimes_R \kappa) = (T - \|v\|^{1-m})(T - \|v\|^{-m})$$

holds in $\kappa[T]$.

Definition 3.6.2. We define $\mathcal{D}^{\mathrm{mix}, \square}$ to be the local lifting condition of $\bar{\rho}$ (see Definition 3.2.3) that classifies liftings of $\bar{\rho}$ to an object R of $\mathcal{C}_{\mathcal{O}}$ such that in the decomposition (3.4), $\rho(I_{F_v})$ preserves M_0 and acts trivially on M_1 . We define

- (1) $\mathcal{D}^{\mathrm{unr}, \square}$ to be the local lifting condition contained in $\mathcal{D}^{\mathrm{mix}, \square}$ such that the action of $\rho(I_{F_v})$ on M_0 is also trivial.
- (2) $\mathcal{D}^{\mathrm{ram}, \square}$ to be the local lifting condition contained in $\mathcal{D}^{\mathrm{mix}, \square}$ such that

$$\det(T - \mathrm{Std} \circ \rho(\phi_v)|M_0) = (T - \|v\|^{1-m})(T - \|v\|^{-m})$$

in $R[T]$.

Proposition 3.6.3. *The formal scheme $\mathcal{D}^{\mathrm{mix}, \square}$ is formally smooth over $\mathrm{Spf} \mathcal{O}[[x_0, x_1]]/(x_0 x_1)$ of pure relative dimension $2m^2 + m - 1$ such that the irreducible components defined by $x_0 = 0$ and $x_1 = 0$ are $\mathcal{D}^{\mathrm{unr}, \square}$ and $\mathcal{D}^{\mathrm{ram}, \square}$, respectively. In particular, $\mathcal{D}^{\mathrm{ram}, \square}$ is formally smooth over $\mathrm{Spf} \mathcal{O}$ of pure relative dimension $2m^2 + m$, and it follows from Equation 3.1 that*

$$\dim_\kappa L(\mathcal{D}_{\bar{\rho}}^{\mathrm{ram}}) = \dim_\kappa H^0(\mathrm{Gal}_{F_v}, \mathrm{ad}^0(\bar{\rho})).$$

Proof. We modify the proof of [LTX⁺24, Proposition 3.35]. Identifying the underlying space of $\mathrm{Std} \circ \bar{\rho}$ with $\kappa^{\oplus(2m)}$, there exists a decomposition

$$\kappa^{\oplus(2m)} = \bar{M}_0 \oplus \bar{M}_1$$

such that

$$\det(T - \mathrm{Std} \circ \bar{\rho}(\phi_v)|\bar{M}_0) = (T - \|v\|^{1-m})(T - \|v\|^{-m})$$

in $\kappa[T]$. Upon changing a basis, we may assume that $\bar{M}_0$ is spanned by the first two factors and $\bar{M}_1$ is spanned by the last $2m - 2$ factors. Then we get two unramified representations $\bar{\rho}_0 : T_{\|v\|} \rightarrow \mathrm{GSp}_2(\kappa)$ and $\bar{\rho}_1 : T_{\|v\|} \rightarrow \mathrm{GSp}_{2m-2}(\kappa)$. Let $\mathcal{D}_0^\square$ be the unrestricted lifting condition of $\bar{\rho}_0$ with respect to $\chi = \varepsilon_\ell^{1-2m}$, and let $\mathcal{D}_1^\square$ be the local lifting condition of $\bar{\rho}_1$ with respect to $\chi = \varepsilon_\ell^{1-2m}$ classifying unramified liftings.

We say that a lifting ρ of $\bar{\rho}$ to an object R of $\mathcal{C}_{\mathcal{O}}$ is *standard* if

$$\rho(t) = \begin{bmatrix} A_0 & 0 \\ 0 & I_{2m-2} \end{bmatrix}, \quad \rho(\phi_v) = \begin{bmatrix} B_0 & 0 \\ 0 & B_1 \end{bmatrix},$$

with $A_0, B_0 \in \mathrm{GL}_2(\mathbf{R})$ and $B_1 \in \mathrm{GSp}_{2m-2}(\mathbf{R})$. Let $\mathcal{D}_{0,1}^{\mathrm{mix},\square} \subset \mathcal{D}^{\mathrm{mix},\square}$ be the subfunctor classifying standard liftings. Then there exists a natural isomorphism

$$\mathcal{D}_{0,1}^{\mathrm{mix},\square} \cong \mathcal{D}_0^{\square} \times_{\mathrm{Spf} \mathcal{O}} \mathcal{D}_1^{\square}.$$

Moreover, there is an isomorphism

$$\mathcal{D}_{0,1}^{\mathrm{mix},\square} \times_{\mathrm{Spf} \mathcal{O}} (\mathrm{Sp}_2^{\wedge} \times_{\mathrm{Spf} \mathcal{O}} \mathrm{Sp}_{2m-2}^{\wedge} \setminus \mathrm{Sp}_{2m}^{\wedge}) \cong \mathcal{D}^{\mathrm{mix},\square}.$$

By Proposition 3.4.6 (here we use that $\ell > 2m$), $\mathcal{D}_1^{\square}$ is formally smooth of relative dimension $(2m-1)(m-1)$ over $\mathrm{Spf} \mathcal{O}$. It follows from [Sho16, Proposition 5.5](2) that $\mathcal{D}_0^{\square}$ is isomorphic to $\mathrm{Spf} \mathcal{O}[[x_0, \dots, x_3]]/(x_0 x_1)$, where the irreducible component defined by $x_0 = 0$ classifies unramified liftings, and the irreducible component defined by $x_1 = 0$ classifies liftings ρ such that $\|v\|^{-m}$ is an eigenvalue of $\mathrm{Std} \circ \rho(\phi_v)$. The assertion follows. $\square$

4. AN ALMOST MINIMAL $\mathbf{R}=\mathbf{T}$ THEOREM

In this section, we prove an almost minimal $\mathbf{R} = \mathbf{T}$ theorem for self-dual Galois representations with coefficients in a finite field satisfying a property called *rigid*.

4.1. Rigid Galois representations. We take a finite set $\Sigma^{\min}$ of finite places of F containing Σ^{bad} . We take an odd rational prime ℓ larger than $2(b_{\xi} - a_{\xi}) + 2$ that is unramified in F , such that $\Sigma^{\min}$ contains no ℓ -adic places. We then work in the setting of Setup 3.0.1.

We fix an isomorphism $\iota_{\ell} : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}_{\ell}}$ and assume that the complex algebraic representation of $\mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_{2m}$ with highest weight ξ can be defined over $\iota_{\ell}^{-1} E$.

We take a finite set Σ^{lr} of finite places of F such that

- $\Sigma^{\mathrm{lr}} = \emptyset$ when N is even;
- Σ^{lr} is disjoint from $\Sigma^{\min} \cup \Sigma(\ell)$; and
- $\|v\|^2 - 1$ is divisible by ℓ for every place $v \in \Sigma^{\mathrm{lr}}$

We fix an element $\xi = (\xi_{\tau})_{\tau} \in (\mathbb{Z}_{\leq}^{2m})^{\Sigma^{\infty}}$ satisfying $\xi_{\tau, i} = -\xi_{\tau, 2m+1-i}$ for every $\tau \in \Sigma^{\infty}$ and $1 \leq i \leq 2m$.

We consider a triple $(\Gamma, \bar{\rho}, \chi)$ from Setup 3.2.1 in which $\Gamma = \mathrm{Gal}_F$ and $\chi = \varepsilon_{\ell}^{2-N}$.

Definition 4.1.1. We say that $\bar{\rho}$ is *rigid* for $(\Sigma^{\min}, \Sigma^{\mathrm{lr}})$ if the following are satisfied:

- (1) For $v \in \Sigma^{\min}$, every lifting of $\bar{\rho}_v$ is minimally ramified (see Definition 3.5.8).
- (2) For $v \in \Sigma^{\mathrm{lr}}$, $\dim_{\kappa} \ker (\bar{\rho}_v(\phi_v) - \|v\|^{-m})^{2m} = 1$.
- (3) For $v \in \Sigma(\ell)$, $\bar{\rho}_v$ is Fontaine–Laffaille crystalline with regular weights in $[a_{\xi}, b_{\xi}]$ (see Definition 3.3.7), such that $\ell > 2(b_{\xi} - a_{\xi}) + 2$.
- (4) For every finite place v of F not in $\Sigma^{\min} \cup \Sigma^{\mathrm{lr}} \cup \Sigma(\ell)$, the homomorphism $\bar{\rho}_v$ is unramified.

Suppose now that $\bar{\rho}$ is rigid for $(\Sigma^{\min}, \Sigma^{\mathrm{lr}})$. Consider a global deformation problem (see Definition 3.2.5)

$$\mathcal{S} := \left(\bar{\rho}, \varepsilon_{\ell}^{2-N}, \Sigma^{\min} \cup \Sigma^{\mathrm{lr}} \cup \Sigma(\ell), \{\mathcal{D}_v^{\square}\}_{v \in \Sigma^{\min} \cup \Sigma^{\mathrm{lr}}, \Sigma(\ell)} \right)$$

where

- for $v \in \Sigma^{\min}$, $\mathcal{D}_v^{\square}$ is the unrestricted local lifting condition of $\bar{\rho}_v$;
- for $v \in \Sigma^{\mathrm{lr}}$, $\mathcal{D}_v^{\square}$ is the local lifting condition $\mathcal{D}_v^{\mathrm{ram}}$ of $\bar{\rho}_v$ from Definition 3.6.2; and
- for $v \in \Sigma(\ell)$, $\mathcal{D}_v^{\square}$ is the local lifting condition $\mathcal{D}_v^{\mathrm{FL}, \square}$ of $\bar{\rho}_v$ from Definition 3.3.8.

Then we have the global deformation ring $\mathbf{R}_{\mathcal{S}}$ from Proposition 3.2.8.

4.2. Adequate subgroups. In this subsection, we recall the notion of G° -adequate subgroups following [Whi23]. We consider a triple $(\Gamma, \bar{\rho}, \chi)$ from Setup 3.2.1 in which $\Gamma = \mathrm{Gal}_F$ and $\chi = \varepsilon_{\ell}^{2-N}$.

Definition 4.2.1. We say that a subgroup $H \leq G^{\circ}(\kappa)$ is G° -adequate if

- (1) the cohomology groups

$$\mathrm{H}^0(H, (\mathfrak{g}'_{\kappa})^{\vee}), \quad \mathrm{H}^1(H, \kappa), \quad \mathrm{H}^1(H, (\mathfrak{g}'_{\kappa})^{\vee})$$

vanish; and

- (2) for every irreducible $\kappa[H]$ -submodule W of $(\mathfrak{g}'_\kappa)^\vee$, there exists an element $h \in H$ such that $w(z) \neq 0$ for some $w \in W$ and some $z \in \text{Lie } Z(Z_{G^\circ}(h)) \cap \mathfrak{g}'_\kappa$.

Here $Z_{G^\circ}(h)$ denotes the centralizer subgroup of h in G° .

Proposition 4.2.2. *If $\ell > 2m + 4$ and $|\ell - 4m| \neq 1$, and Γ' is a subgroup of Γ such that $\bar{\rho}(\Gamma') \subset G^\circ(\kappa)$ and $\bar{\rho}|_{\Gamma'}$ is absolutely irreducible, then $\bar{\rho}(\Gamma')$ is a G° -adequate subgroup of $G^\circ(\kappa)$.*

Proof. This follows from [Whi23, Theorem 2.3.4]. $\square$

4.3. Finiteness of deformation rings. In this subsection, we prove a finiteness theorem for the deformation ring of a self-dual Galois representation that will be used in §5.2. Let Π be a $\mathfrak{d}$ -REASDC representation of $\text{GL}_{2m}(\mathbf{A}_F)$ (see Definition 2.2.2). Let $\Sigma^{\min}$ be a finite set of finite places of F containing Σ^{bad} such that Π_v is unramified for every finite place v of F not in $\Sigma^{\min}$. Let $E \subset \mathbb{C}$ be a strong coefficient field of Π (see Definition 2.2.5). Let λ be a finite place of E whose underlying rational prime ℓ satisfies

$$\Sigma(\ell) \cap \Sigma^{\min} = \emptyset, \quad \ell > 2(2m + 1), \quad \ell > 2(b_\xi - a_\xi) + 2,$$

and $\ell > \|v\|^{2m}$ for every place v in $\Sigma^{\min}$. We set $\Sigma := \Sigma^{\min} \cup \Sigma(\ell)$.

Then we adopt the notation of Setup 3.0.1 with $E = E_\lambda$ plus the notation of Setup 3.2.1 with $(\Gamma, \bar{\rho}, \chi) = (\text{Gal}_F, \bar{\rho}_{\Pi, \lambda}, \varepsilon_\ell^{1-2m})$, and fix an isomorphism $\iota_\ell : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_\ell$. Let

$$\mathcal{S} := \left(\bar{\rho}_{\Pi, \lambda}, \varepsilon_\ell^{2-N}, \Sigma^0, \{\mathcal{D}_v^\square\}_{v \in \Sigma} \right)$$

be a global deformation problem (see Definition 3.2.5), where for each $v \in \Sigma^{\min}$, $\mathcal{D}_v^\square$ is an irreducible component (which is a local deformation problem by Proposition 3.5.11(2)); and for each $v \in \Sigma(\ell)$, $\mathcal{D}_v^\square$ is the local deformation $\mathcal{D}^{\text{FL}, \square}$ of $\bar{\rho}_{\Pi, \lambda}$ from Definition 3.3.8.

Definition 4.3.1. Let $\mathcal{G}_{2m}$ be the group scheme over $\mathbb{Z}$ which is the semidirect product of the group $\text{GL}_{2m} \times \text{GL}_1$ by the group $\{1, j\}$, which acts on $\text{GL}_{2m} \times \text{GL}_1$ by

$$j(g, \mu)j^{-1} = (\mu g^{-\top}, \mu).$$

There is a homomorphism $\nu_{\mathcal{G}} : \mathcal{G}_{2m} \rightarrow \text{GL}_1$ sending (g, μ) to μ and j to -1 .

Let $\tilde{F}$ be a totally imaginary quadratic extension of F that is linearly disjoint from $(\overline{F})^{\ker \bar{\rho}(\mu_\ell)}$. We fix a real place of F that induces a complex multiplication $\mathfrak{c} \in \text{Gal}_F$. We also choose an algebraic character

$$\psi : \text{Gal}(\overline{F}/\tilde{F}) \rightarrow \overline{\mathbb{Q}}_\ell^\times$$

that is crystalline at places of F above ℓ , satisfying $\psi\psi^c = \varepsilon_\ell$ (resp. ψ is trivial) when N is even (resp. N is odd). This is possible by [CHT08, Lemma 4.1.5]. For our purpose of proving the finiteness of the deformation ring $R_{\mathcal{S}}$, we can enlarge L so that ψ is valued in $\mathcal{O}_L^\times$.

Recall that, for any object R of $\mathcal{C}$ and any continuous homomorphism $\rho' : \text{Gal}_F \rightarrow G(R)$ with $\nu \circ \rho' = \varepsilon_\ell^{2-N}$, there is a continuous homomorphism $r' : \text{Gal}_F \rightarrow \mathcal{G}_{2m}(R)$, determined by

$$r'(g) = \begin{cases} (\psi^{-1}(g)\rho'(g), \varepsilon_\ell^{1-2m}(g)) & g \in \text{Gal}_{\tilde{F}} \\ (\psi^{-1}(gc_v)\rho'(g)(J'_{2m})^{-1}, -\varepsilon_\ell^{1-2m}(g))j & g \notin \text{Gal}_{\tilde{F}} \end{cases},$$

cf. [CHT08, Lemma 2.1.2]. Moreover,

$$\nu_{\mathcal{G}} \circ r' = \varepsilon_\ell^{1-2m}.$$

This construction is compatible with equivalences in the sense that if $B \in (G')^\wedge(R)$ and ρ' is replaced by $\rho'_B := B\rho'B^{-1}$, then r' is replaced by $r'_B := (a \text{Std}(B), 1)r'(a \text{Std}(B), 1)^{-1}$, where $a \in R$ satisfies $a^2 = \nu(B)^{-1}$.

Theorem 4.3.2. *Suppose that $\text{Std} \circ \bar{\rho}_{\Pi, \lambda}|_{\text{Gal}_F(\mu_\ell)}$ is absolutely irreducible. Then $R_{\mathcal{S}}$ is a finite $\mathcal{O}$ -module.*

Proof. Fix a lift $\rho_{\mathcal{S}}$ of $\bar{\rho}$ to $R_{\mathcal{S}}$ representing the universal deformation. From the construction above, we obtain a homomorphism

$$r_{\mathcal{S}} : \text{Gal}_F \rightarrow \mathcal{G}_{2m}(R_{\mathcal{S}}).$$

We may consider the corresponding residual representation

$$\bar{r}_{\mathcal{S}} : \text{Gal}_F \rightarrow \mathcal{G}_{2m}(\kappa)$$

and for each $v \in \Sigma^{\min}$ a local deformation problem $\tilde{\mathcal{D}}_v^\square$ of $\bar{r}_{\mathcal{F}}$ corresponding to an irreducible component of the unrestricted deformation problem (cf. [LTX⁺24, Proposition 3.33]) such that $r_{\mathcal{F}}|_{\mathrm{Gal}_{F_v}}$ is an object of $\tilde{\mathcal{D}}_v^\square(\mathcal{R}_{\mathcal{F}})$. We consider (in the notation of [LTX⁺24, Definition 3.6]) the deformation problem

$$\tilde{\mathcal{F}} := (\bar{r}_{\mathcal{F}}, \varepsilon_\ell^{1-2m}, \Sigma, \{\tilde{\mathcal{D}}_v^\square\}_{v \in \Sigma}),$$

$\tilde{\mathcal{D}}_v^\square$ is the Fontaine–Laffaille deformation problem as defined in [LTX⁺24, Definition 3.12] for each $v \in \Sigma(\ell)$. Fix a universal lift

$$r_{\tilde{\mathcal{F}}} : \mathrm{Gal}_F \rightarrow \mathcal{G}_{2m}(\mathcal{R}_{\tilde{\mathcal{F}}}).$$

Then there exists a homomorphism $\theta : \mathcal{R}_{\tilde{\mathcal{F}}} \rightarrow \mathcal{R}_{\mathcal{F}}$ such that $\theta \circ r_{\tilde{\mathcal{F}}}$ is equivalent to $r_{\mathcal{F}}$.

We claim that θ makes $\mathcal{R}_{\mathcal{F}}$ a finite module over $\mathcal{R}_{\tilde{\mathcal{F}}}$. To show this, we denote $\mathcal{R}_{\tilde{\mathcal{F}}, \mathcal{F}} := \mathcal{R}_{\mathcal{F}} / \theta(\mathfrak{m}_{\mathcal{R}_{\tilde{\mathcal{F}}}})$, and let $\rho_{\tilde{F}, F}$ denote the $G(\mathcal{R}_{\tilde{\mathcal{F}}, \mathcal{F}})$ -valued representation obtained from $\theta_{\mathcal{F}}$, and let

$$r_{\tilde{\mathcal{F}}, \mathcal{F}} : \mathrm{Gal}_F \rightarrow \mathcal{G}_{2m}(\mathcal{R}_{\tilde{\mathcal{F}}, \mathcal{F}})$$

denote the corresponding homomorphism. Then $r_{\tilde{\mathcal{F}}, \mathcal{F}}$ is equivalent to $\bar{r}_{\mathcal{F}}$. In particular, $r_{\tilde{\mathcal{F}}, \mathcal{F}}$ has finite image. It follows from Lemma 3.1.3 that $\mathcal{R}_{\mathcal{F}}$ is generated by the traces and similitudes of images of $\mathrm{Std} \circ \rho_{\mathcal{F}}$. Consider any prime ideal $\mathfrak{p}$ of $\mathcal{R}_{\tilde{\mathcal{F}}, \mathcal{F}}$. Because the image of $r_{\tilde{\mathcal{F}}, \mathcal{F}}$ is finite, we see that the images of these traces in $\mathcal{R}_{\tilde{\mathcal{F}}, \mathcal{F}}/\mathfrak{p}$ are sums of roots of unity of bounded degree (up to a scalar in $\mathcal{O}^\times$). In particular, $\mathcal{R}_{\tilde{\mathcal{F}}, \mathcal{F}}/\mathfrak{p}$ is finite. Thus $\mathcal{R}_{\tilde{\mathcal{F}}, \mathcal{F}}$ is Artinian. Because it has finite residue field, it is itself finite. The claim then follows from Nakayama’s lemma.

Note that $\mathrm{Std} \circ \bar{\rho}_{\Pi, \lambda}|_{\mathrm{Gal}(\bar{F}/F(\mu_\ell))}$ is absolutely irreducible by our hypothesis on $\tilde{F}$; which implies $\mathrm{Std} \circ \bar{\rho}_{\Pi, \lambda}(\mathrm{Gal}(\bar{F}/\tilde{F}(\mu_\ell)))$ is adequate in the sense of [Tho12, Definition 2.3] by [Tho12, Theorem A.9]. By the same proof of [Gee11, Theorem 5.1.4] (strengthened using Thorne’s adequacy condition [Tho12]), we know that $\mathcal{R}_{\tilde{\mathcal{F}}}$ is a finite $\mathcal{O}$ -module.

The theorem is proved. $\square$

4.4. An almost minimal $\mathbf{R}=\mathbf{T}$ theorem. In this subsection, we work in the following setting:

Setup 4.4.1.

- Let $\mathbf{V}$ be a quadratic space over F of rank N and discriminant $\mathfrak{d}$, such that $\mathbf{V}_v$ is not split for $v \in \Sigma^{\mathrm{lr}}$. Let $(p_\tau, q_\tau)_{\tau \in \Sigma^\infty}$ be the signature of $\mathbf{V}$. Suppose that $q_\tau \in \{0, 2\}$ for each $\tau \in \Sigma^\infty$, and set $d(\mathbf{V}) := (\sum_{\tau \in \Sigma^\infty} p_\tau q_\tau)/2$.
- Let Λ be a self-dual $\prod_{v \in \Sigma^\infty \cup \Sigma^{\min}} \mathcal{O}_v$ -lattice in $\mathbf{V} \otimes_F \mathbf{A}_F^{\Sigma^\infty \cup \Sigma^{\min} \cup \Sigma^{\mathrm{lr}}}$ and a neat compact open subgroup of the form

$$\mathcal{K} = \prod_{v \in \Sigma^{\min} \cup \Sigma^{\mathrm{lr}}} \mathcal{K}_v \times \prod_{v \notin \Sigma^\infty \cup \Sigma^{\min} \cup \Sigma^{\mathrm{lr}}} \mathrm{SO}(\Lambda)(\mathcal{O}_v)$$

in which $\mathcal{K}_v$ is special maximal for $v \in \Sigma^{\mathrm{lr}}$.

- Let Σ be a finite set of finite places of F containing $\Sigma^{\min} \cup \Sigma^{\mathrm{lr}} \cup \Sigma(\ell)$.

We obtain a system of (complex) Shimura varieties $\{\mathrm{Sh}(\mathbf{V}, \mathcal{K}')\}_{\mathcal{K}'}$ attached to $\mathrm{Res}_{F/\mathbb{Q}} \mathrm{SO}(\mathbf{V})$ indexed by compact open subgroups $\mathcal{K}'$ of $\mathcal{K}$, which are quasi-projective smooth complex schemes of dimension $d(\mathbf{V})$. The element ξ gives rise to a continuous homomorphism

$$\prod_{v \in \Sigma(\ell)} \mathrm{SO}(\Lambda)(\mathcal{O}_v) \rightarrow \mathrm{GL}(L_\xi),$$

where L_ξ is a finite free $\mathcal{O}$ -module, hence induces an $\mathcal{O}$ -linear étale local system $\mathcal{L}_\xi$ on $\mathrm{Sh}(\mathbf{V}, \mathcal{K}')$ for every compact open subgroups $\mathcal{K}'$ of $\mathcal{K}$, compatible under pullback maps; see [KSZ21, 1.5.8].

Recall the abstract orthogonal Hecke algebra $\mathbb{T}^{m, \mathfrak{d}, \Sigma}$ defined in Definition 2.1.3. Let $\phi : \mathbb{T}^{m, \mathfrak{d}, \Sigma} \rightarrow \kappa$ be the homomorphism such that for every finite place v of F not in Σ , we have $\phi|_{\mathbb{T}_v^{m, \mathfrak{d}}} = \phi_{\alpha_v}$ (see Definition 2.1.4) where $\alpha_v = (\alpha_{v,1}^{\pm 1}, \dots, \alpha_{v,m}^{\pm 1})$ is the orthogonal abstract Satake parameter at v (see Definition 2.1.2) satisfying that

$$\left\{ \alpha_{v,1}^{\pm 1} \iota_\ell(\sqrt{\|v\|})^{2-N}, \dots, \alpha_{v,m}^{\pm 1} \iota_\ell(\sqrt{\|v\|})^{2-N} \right\}$$

is the multiset of generalized eigenvalues of $\bar{\rho}_v(\phi_v)$ in $\overline{\mathbb{F}_\ell}$. We denote by $\mathfrak{m}$ the kernel of ϕ .

Theorem 4.4.2. *Under the above setup, we assume*

(D0) (already assumed) $\ell > 2(b_\xi - a_\xi) + 2$, and ℓ is unramified in F ;

(D1) $\ell > 2(2m + 1)$;

(D2) $\text{Std} \circ \bar{\rho}|_{\text{Gal}(\bar{F}/F(\sqrt{\mathfrak{d}})(\mu_\ell))}$ is absolutely irreducible;

(D3) $\bar{\rho}$ is rigid for $(\Sigma^{\min}, \Sigma^{\text{lr}})$;

(D4) for every finite set Σ' of finite places of F containing Σ , and every compact open group $\mathcal{K}'$ of $\mathcal{K}$ of the form $\mathcal{K}' = \mathcal{K}'_{\Sigma'} \times \mathcal{K}^{\Sigma'}$, we have

$$H_{\text{ét}}^d(\text{Sh}(\mathbf{V}, \mathcal{K}'), \mathcal{L}_\xi \otimes_{\mathcal{O}} \kappa)_{\mathbb{T}_m, \mathfrak{d}, \Sigma' \cap m} = 0$$

unless $d = d(\mathbf{V})$.

Let $\mathbb{T}$ be the image of $\mathbb{T}^{m, \mathfrak{d}, \Sigma}$ in $\text{End}_{\mathcal{O}}(H_{\text{ét}}^{d(\mathbf{V})}(\text{Sh}(\mathbf{V}, \mathcal{K}), \mathcal{L}_\xi))$. If $\mathbb{T}_m \neq 0$, then

(1) There is a canonical isomorphism $R_{\mathcal{S}} \cong \mathbb{T}_m$ of local complete intersection rings over $\mathcal{O}$.

(2) The $\mathbb{T}_m$ -module $H_{\text{ét}}^{d(\mathbf{V})}(\text{Sh}(\mathbf{V}, \mathcal{K}), \mathcal{L}_\xi)_m$ is finite and free.

Remark 4.4.3. The Condition (D4) is assumed because we need to use torsion-freeness of the cohomology with coefficients in the local system $\mathcal{L}_\xi$. When $\mathcal{L}_{\xi, \lambda}$ is a constant local system, this condition can be dropped by result of the author [Pen25a] when $\text{Sh}(\mathbf{V}, \mathcal{K})$ is proper, and by Yang–Zhu [YZ25] in general.

The rest of this section is devoted to the proof of the theorem. We modify the proof of [LTX⁺24, Theorem 3.38]. By definition, we have an identification

$$\mathcal{H}_{\mathcal{K}_0(\emptyset), m} := \text{Hom}_{\mathcal{O}}(H_{\text{ét}}^{d(\mathbf{V})}(\text{Sh}(\mathbf{V}, \mathcal{K}_0(\emptyset)), \mathcal{L}_\xi)_m, \mathcal{O}),$$

under which $\mathbb{T}_m$ is identified with the image of $\mathbb{T}^{m, \mathfrak{d}, \Sigma}$ in $\text{End}(\mathcal{H}_{\mathcal{K}_0(\emptyset), m})$ since $H_{\text{ét}}^{d(\mathbf{V})}(\text{Sh}(\mathbf{V}, \mathcal{K}), \mathcal{L}_\xi)_m$ is $\mathcal{O}$ -torsion free.

We first construct a canonical homomorphism $R_{\mathcal{S}} \rightarrow \mathbb{T}_m$. It follows from semisimplicity of Hecke operators that $\mathbb{T}_m[\frac{1}{\ell}]$ is a reduced ring that is finite over E . As $\mathcal{H}_{\mathcal{K}_0(\emptyset), m}$ is a finite free $\mathcal{O}$ -module, $\mathbb{T}_m$ is also finite free over $\mathcal{O}$. Via ι_ℓ^{-1} , every maximal ideal x of $\mathbb{T}_m[\frac{1}{\ell}]$ gives rise to an $\mathfrak{d}$ -REASDC representation Π_x of $\text{GL}_{2m}(\mathbf{A}_F)$ (see Definition 2.2.2), satisfying that

- (a) the associated Galois representation ρ_{Π_x, ι_ℓ} from Proposition 2.2.3 is residually isomorphic to $\bar{\rho} \otimes_{\kappa} \bar{\mathbb{F}}_\ell$;
- (b) Π_x is the functorial lifting (see Definition 2.3.1) of a cuspidal automorphic representation π of $\text{SO}(\mathbf{V})(\mathbf{A}_F)$ satisfying that $(\pi^\infty)^{\mathcal{K}}$ appears nontrivially in $H_{\text{ét}}^{d(\mathbf{V})}(\text{Sh}(\mathbf{V}, \mathcal{K}), \mathcal{L}_\xi \otimes_{\mathcal{O}, \iota_\ell^{-1}} \mathbb{C})$. In particular, the Archimedean weights of Π equals ξ (see Definition 2.2.2). Note that Π_x is cuspidal because it is associated to an irreducible Galois representation.

Denote by $\mathbb{T}_x$ the localization of $\mathbb{T}_m[\frac{1}{\ell}]$ at x . By an argument using Baire category theorem as in [CHT08, Lemma 2.1.5], we may assume that the Galois representation ρ_{Π_x, ι_ℓ} is actually valued in $G(\mathcal{O}_x)$, where $\mathcal{O}_x$ is the ring of integers of some finite extension of $\mathbb{T}_m[\frac{1}{\ell}]/x$. We may also assume that the reduction of ρ_x modulo the maximal ideal of $\mathcal{O}_x$ equals $\bar{\rho}$. (Not just conjugate to $\bar{\rho}$). Let A denote the subring of $\kappa \oplus \bigoplus_{x \in \text{Spec } \mathbb{T}_m[\frac{1}{\ell}]} \mathcal{O}_x$ consisting of elements (a_m, a_x) such that the reduction of a_x modulo the maximal ideal of $\mathcal{O}_x$ is a_m for all $x \in \text{Spec } \mathbb{T}_m[\frac{1}{\ell}]$. Then we have a continuous homomorphism

$$\bar{\rho} \oplus \bigoplus_{x \in \text{Spec } \mathbb{T}_m[\frac{1}{\ell}]} \rho_x : \text{Gal}_F \rightarrow G(A).$$

It follows from Lemma 3.1.3 that $\bar{\rho} \oplus \bigoplus_{x \in \text{Spec } \mathbb{T}_m[\frac{1}{\ell}]} \rho_x$ is conjugate to a continuous homomorphism

$$\rho_m : \text{Gal}_F \rightarrow G(\mathbb{T}_m)$$

that lifts $\bar{\rho}$.

We claim that ρ_m satisfies the global deformation problem $\mathcal{S}$. By property (b) and Arthur’s multiplicity formula, $\Pi_{x, v}$ is unramified for every finite place v of F not in $\Sigma^{\min} \cup \Sigma^{\text{lr}}$. So it follows from property (b) and Proposition 2.2.3(2) that $\rho_{m, v}$ is unramified for every v not in Σ . It follows from property (b), assumption (D0) and Proposition 2.2.3(3) that $\rho_{m, v}$ belongs to $\mathcal{D}_v^{\text{FL}, \square}$ for every $v \in \Sigma(\ell)$. It follows from property (a)–(b), Proposition 2.2.3(2), Corollary 2.4.2 and Arthur’s multiplicity formula that $\rho_{m, v}$ belongs to $\mathcal{D}_v^{\text{ram}, \square}$ for every $v \in \Sigma^{\text{lr}}$.

Therefore, by the universal property of $R_{\mathcal{S}}$, we obtain a canonical homomorphism

$$\varphi : R_{\mathcal{S}} \rightarrow \mathbb{T}_{\mathfrak{m}}$$

of rings over $\mathcal{O}$.

We now prove that φ is surjective. For this, we may assume that E contains $\mathbb{Q}_{\ell^2}$ and every algebraic representation of G is defined over E , so $\|v\|$ is a square in $E^\times$ for every finite place of F not in Σ . Fix a lift $\rho_{\mathcal{S}}$ of $\bar{\rho}$ to $R_{\mathcal{S}}$. For each algebraic $\mathcal{O}$ -representation (r_V, V) of G and each finite place v of F not in Σ , define

$$t_{v,V}^R := \text{tr}(r_V(\rho_{\mathcal{S}}(\phi_v))) \in R_{\mathcal{S}}$$

and

$$t_{v,V}^T := \text{tr}(r_V(\rho_{\mathfrak{m}}(\phi_v))) \in \mathbb{T}_{\mathfrak{m}}.$$

By construction of ϕ , we have

$$\varphi(t_{v,V}^R) = t_{v,V}^T.$$

Thus it follows from the integral Satake isomorphism (cf. [Zhu24]) that $\mathbb{T}_{\mathfrak{m}}$ is generated by the elements $t_{v,V}^T$. In particular, φ is surjective.

In the next two subsections, we will use the Taylor–Wiles patching argument to prove that φ is injective.

4.5. Local computations. In this subsection, we study local Langlands correspondence for principal series representations of split p -adic odd special orthogonal groups. We work in the following setting

Setup 4.5.1.

- Let v be a finite place of F .
- Let SO_N be the split orthogonal group of rank m over $\mathcal{O}_v$.
- Let T^{SO} be a split maximal torus of SO_N . Let B^{SO} be a Borel subgroup containing T^{SO} , with unipotent radical U_0^{SO} . Let W be the Weyl group of SO_N .
- Let ℓ be an odd rational prime not dividing the order of W , and suppose that $\|v\| \equiv 1 \pmod{\ell}$.
- Let T_{κ} be a split maximal torus contained in a Borel subgroup B_{κ} of G_{κ}° . Suppose the induced pinned root datum is dual to that of SO_N , and let ι denote the induced isomorphisms $X_*(T^{\text{SO}}) \xrightarrow{\sim} X^*(T_{\kappa})$ and $X^*(T^{\text{SO}}) \xrightarrow{\sim} X_*(T_{\kappa})$, cf. [Whi23, §2.2.3].
- Let $\bar{g}$ be an element of $T_{\kappa}(\kappa)$. Then we can define the following objects:
 - Let $M_{\bar{g}} := Z_{G_{\kappa}^{\circ}}(\bar{g})$ be the centralizer of $\bar{g}$ in G_{κ}° .
 - Let S_{κ} be the identity component of the center of $M_{\bar{g}}$, which is a subtorus of T_{κ} .
 - Let $L_{\kappa} := Z_{G_{\kappa}^{\circ}}(S_{\kappa})$ be the centralizer of S_{κ} in G_{κ}° , which is a Levi subgroup containing $M_{\bar{g}}$.
 - Let P_{κ} denote the standard parabolic subgroup of G_{κ}° admitting L_{κ} as a Levi factor.
 - Let P^{SO} be a standard parabolic subgroup of $\text{SO}_N \otimes \kappa_v$, with unipotent radical U^{SO} and a standard Levi subgroup L^{SO} , such that P^{SO} (resp. L^{SO}) corresponds to $P_{\kappa} \cap G'_{\kappa}$ (resp. $L_{\kappa} \cap G'_{\kappa}$) via ι , cf. [Whi23, §2.2.3]. Let $W_{L^{\text{SO}}}$ be the Weyl group of L^{SO} .
 - let $\mathcal{K}_0$ be the parahoric subgroup of $\text{SO}_N(\mathcal{O}_v)$ consisting of elements whose reduction is in $P^{\text{SO}}(\kappa_v)$.
 - Let Δ be the maximal quotient of $(L^{\text{SO}}/[L^{\text{SO}}, L^{\text{SO}}])(\kappa_v)$ of ℓ -power order.
 - Let $\mathcal{K}_1$ be the kernel of the composition map

$$\mathcal{K}_0 \rightarrow P^{\text{SO}}(\kappa) \rightarrow L^{\text{SO}}(\kappa) \rightarrow \Delta,$$

which is a normal subgroup of $\mathcal{K}_0$.

- Let $\bar{\chi} : X_*(T^{\text{SO}}) \rightarrow \kappa^\times$ be the character corresponding to $\bar{g}$ via ι .
- Let $\mathfrak{n}_1$ (resp. $\mathfrak{n}_0$) denote the maximal ideal of $\mathcal{O}[T^{\text{SO}}(F_v)/(T^{\text{SO}}(\mathcal{O}_v) \cap \mathcal{K}_1)]^{W_{L^{\text{SO}}}}$ (resp. $\mathcal{O}[T^{\text{SO}}(F_v)/T^{\text{SO}}(\mathcal{O}_v)]^{W_{L^{\text{SO}}}}$) corresponding to $\bar{\chi}$.

Suppose (U_0^{SO}, ψ) is a Whittaker datum, that is, ψ is a nondegenerate character of U_0^{SO} . Let $\text{Irr}(\text{SO}_N, T^{\text{SO}})$ denote the set of irreducible smooth representations of $\text{SO}_N(F_v)$ in the principal series. Let $\Phi_e(\text{SO}_N, T^{\text{SO}})$ denote the set of enhanced L -parameters for SO_N associated with principal series, that is, enhanced L -parameters whose cuspidal support is an enhanced L -parameter for T^{SO} ; cf. [Sol25, p.271].

In [Sol25], Solleveld uses the Whittaker datum (U_0^{SO}, ψ) to construct a canonical (bijective) local Langlands correspondence

$$\text{Irr}(\text{SO}_N, T^{\text{SO}}) \leftrightarrow \Phi_e(\text{SO}_N, T^{\text{SO}}) : \quad \pi \mapsto (\phi_\pi, \rho_\pi), \quad \pi(\phi, \rho) \leftarrow (\phi, \rho).$$

well-defined up to G' -conjugacy.

Proposition 4.5.2. *The local Langlands correspondence defined by Solleveld in [Sol25] is compatible with that of Arthur for principal series representations of $\text{SO}_N(F_v)$.*

Proof. We use [Sol25, Lemma 7.13]. In particular, it is known that the local Langlands correspondence

$$\pi \mapsto \phi_\pi$$

of Solleveld is uniquely determined by the following properties:

- (1) This bijection is equivariant for the canonical actions of $H^1(W_{F_v}, Z(G'))$.
- (2) This bijection is compatible with the cuspidal support maps on both sides in the sense of [Sol25, Lemma 7.6].
- (3) Under this bijection, $\pi(\phi, \rho)$ is tempered if and only if ϕ is bounded.
- (4) The bijection is compatible with the Langlands classification and (for tempered representations) with parabolic induction, in the sense of [Sol25, Lemma 7.11].

Thus it suffices to show that the local Langlands correspondence defined by Arthur in [Art13] (and refined in [Pen25a] when N is even) satisfies these properties.

Property (1) holds because Arthur's packets are characterized by endoscopic character identities, and the action of $H^1(W_{F_v}, Z(G'))$ preserves these identities.

Property (2–4) follow from [Pen25a, Theorem 7.1.1] for N even and [AG17, Theorem 3.2] for N odd. $\square$

Proposition 4.5.3. *Let π be a smooth irreducible representation of $\text{SO}_N(F_v)$. Suppose that $(\pi^{\mathfrak{p}_1})_{\mathfrak{n}_1} \neq 0$. Let (r, N) be the Weil–Deligne representation corresponding to π under the local Langlands correspondence of Solleveld [Sol25]. Then $N = 0$.*

Proof. The argument is analogous to [Whi23, Theorem 2.4.26], where a similar result is established with the local Langlands correspondence of [Sol25] replaced by that of [ABPS17].

To prove this claim, recall that the local Langlands correspondence in [Sol25] is obtained on each principal series Bernstein component $\mathfrak{s} = [T^{\text{SO}}, \chi_0]_{\text{SO}_N}$ via a series of canonical bijections

$$\text{Irr}(\text{SO}_N, T^{\text{SO}})^{\mathfrak{s}} \leftrightarrow \text{Irr}(\text{End}_{\text{SO}_N}(\Pi_{\mathfrak{s}})^{\text{op}}) \leftrightarrow \text{Irr}(\mathcal{H}(\mathfrak{s})^{\text{op}}) \leftrightarrow \text{Irr}(\mathcal{H}(\mathfrak{s}^\vee, \|v\|^{1/2})) \leftrightarrow \Phi_e(\text{SO}_N)^{\mathfrak{s}^\vee};$$

cf. [Sol25, §7]. Here

- $\mathfrak{s}^\vee$ is the dual Bernstein component in $\Phi_e(T^{\text{SO}})$ associated with $\mathfrak{s}$ via the local Langlands correspondence for tori; cf. [Sol25, §5];
- $\Pi_{\mathfrak{s}}$ is a progenerator of the Bernstein block $\text{Irr}(\text{SO}_N, T^{\text{SO}})^{\mathfrak{s}}$ in the principal series, given by

$$\Pi_{\mathfrak{s}} := I_{B^{\text{SO}}}^{\text{SO}_N}(\text{ind}_{T_{\text{cpct}}^{\text{SO}}(F_v)}^{T^{\text{SO}}(F_v)}(\chi_c)),$$

where $T_{\text{cpct}}^{\text{SO}}$ is the unique maximal compact subgroup of $T^{\text{SO}}(F_v)$, and $\chi_c := \chi_0|_{T_{\text{cpct}}^{\text{SO}}}$;

- $\mathcal{H}(\mathfrak{s}) = \mathcal{H}(\mathfrak{s})^\circ \rtimes \Gamma_{\mathfrak{s}}$ is certain (extended) affine Hecke algebra that is (canonically when given the Whittaker datum) isomorphic to $\text{End}_{\text{SO}_N}(\Pi_{\mathfrak{s}})$, cf. [Sol25, Theorem 2.7]; and
- $(\mathcal{H}(\mathfrak{s}^\vee, \|v\|^{1/2}))$ is certain (extended) affine Hecke algebra from [AMS24] attached to the Bernstein component $\Phi_e(\text{SO}_N)^{\mathfrak{s}^\vee}$.

The affine Hecke algebra $\mathcal{H}(\mathfrak{s})$ contains a finite-dimensional Iwahori–Hecke subalgebra

$$\mathcal{H}(W(R_{\mathfrak{s}}^\vee), \|v\|^\lambda),$$

where $R_{\mathfrak{s}}^\vee$ is the root datum attached to the Bernstein component $\mathfrak{s}$ and $\lambda : R_{\mathfrak{s}}^\vee/W_{\mathfrak{s}} \rightarrow \mathbb{N}$ is the label function; cf. [Sol25, p.272].

By [Whi23, Proposition 2.4.22], π is contained in a principal series Bernstein component $\mathfrak{s}$. By [Whi23, Lemma 2.4.25], the $\mathcal{H}(W(R_{\mathfrak{s}}^\vee), \|v\|^\lambda)$ -module $\mathcal{M}_\pi$ corresponding to π contains the one-dimensional character on which each simple generator T_{s_α} acts by $\|v\|$ (i.e., the $\|v\|$ -trivial character).

Thus it follows from [ABPS17, Proposition 10.1] that the Kazhdan–Lusztig triple (σ, x, ρ°) attached to $\mathcal{M}_\pi$ satisfies $x = 1$.

Fix an element u in the maximal compact real subtorus of $T_{\mathfrak{s}^\vee}$. Using graded Hecke algebras, Solleveld attached to $\mathcal{M}_\pi$ a triple (σ, y, ρ°) in which

(1) $\sigma \in \text{Lie}(G'_u)$ is semisimple,

(2) $y \in \text{Lie}(G'_u)$ is unipotent,

(3) $[\sigma, y] = -\log(\|v\|)y$, and

(4) ρ° is an irreducible representation of $\pi_0(Z_{G'_u}/Z(G'_u))$ appearing in certain Springer-type homology; cf. [Sol25, p.310]. It follows from the proof of [Sol25, Proposition 6.5] that $(\exp(\sigma), \exp(y), \rho^\circ)$ is the Kazhdan–Lusztig triple associated to $\mathcal{M}_\pi$. In particular, $y = 0$.

On the other hand, it follows from the proof of [Sol25, Lemma 6.10] on page 316 that

$$y = \log \phi_\pi(1, \begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix}).$$

Thus ϕ_π is trivial on the SL_2 -factor, and the assertion follows. $\square$

Combining Proposition 4.5.3 and Proposition 4.5.2, we get the following corollary.

Corollary 4.5.4. *Let π be a smooth irreducible representation of $\text{SO}_N(F_v)$. Suppose that $(\pi^{\mathfrak{p}_1})_{\mathfrak{n}_1} \neq 0$. Let (r, N) be the Weil–Deligne representation corresponding to π under the local Langlands correspondence of Solleveld [Sol25]. Then $N = 0$.*

4.6. Taylor–Wiles patching argument. We will use the Taylor–Wiles patching argument following [CHT08], [Tho12], and [Whi23]. Set $\Sigma^0 := \Sigma^{\min} \cup \Sigma^{\text{lr}} \cup \Sigma(\ell)$. To continue the proof of Theorem 4.4.2, we may replace E by a finite unramified extension in $\overline{\mathbb{Q}_\ell}$. Thus we may assume that κ contains all eigenvalues of matrices in $\overline{\rho}(\text{Gal}_F)$. For each compact open subset $\mathcal{K}'$ of $\mathcal{K}$, we set

$$\mathcal{H}_{\mathcal{K}'} := \text{Hom}_{\mathcal{O}} \left(H_{\text{ét}}^{d(\mathbf{V})}(\text{Sh}(\mathbf{V}, \mathcal{K}'), \mathcal{L}_\xi), \mathcal{O} \right).$$

We recall the definition of Taylor–Wiles places and Taylor–Wiles data, cf. [Whi23, Definitions 2.2.8–2.2.9].

Definition 4.6.1. A *Taylor–Wiles place* for the global deformation problem $\mathcal{S}$ is a finite place v of F not in Σ that is split in $F(\sqrt{\mathfrak{d}})$, such that $\|v\| \equiv 1 \pmod{\ell}$ and $\overline{g}_v := \overline{\rho}(\phi_v) \in G^\circ(\kappa)$ is semisimple. Such a place is said to be of level $n \geq 1$ if $\|v\| \equiv 1 \pmod{\ell^n}$.

If v is a Taylor–Wiles place for $\mathcal{S}$, we let $M_{\overline{g}_v} := Z_{G_\kappa^\circ}(\overline{g}_v)$ denote the centralizer of $\overline{g}_v$ in G_κ° .

A *Taylor–Wiles datum* is a pair

$$(\Sigma^{\text{TW}}, \{(T_v, B_v)\}_{v \in \Sigma^{\text{TW}}})$$

in which

(1) Σ^{TW} is a finite set of Taylor–Wiles places of F ; and

(2) for each $v \in \Sigma^{\text{TW}}$, $T_v \leq G_\kappa^\circ$ is a split maximal torus containing $Z(M_{\overline{g}_v})^\circ$, and B_v is a Borel subgroup of G_κ° containing T_v .

Proposition 4.6.2. *For each Taylor–Wiles place $v \in \Sigma^{\text{TW}}$, there is a local deformation condition $\mathcal{D}_v^\square$ of $\overline{\rho}_v$ (with respect to $\chi = \varepsilon_\ell^{2-N}$) representing liftings ρ_v of $\overline{\rho}_v$ to objects A of $\mathcal{C}_{\mathcal{O}}$ satisfying $\rho_v(I_v) \subset Z(M_{g_v})^\circ(A)$. Here $g_v = \rho_v(\phi_v)$ and M_{g_v} is the subgroup constructed in [Whi23, Theorem 2.1.14], which is a lift of $M_{\overline{g}_v}$.*

Moreover, we have

$$\dim_\kappa L(\mathcal{D}_v^\square) = \dim_\kappa H^0(F_v, \text{ad}^0(\overline{\rho})) + \dim_\kappa Z(M_{\overline{\rho}(\phi_v)}) - 1.$$

Proof. This follows from [Whi23, Definition 2.2.10 and Corollary 2.2.13] applied to G° . The last equation follows from the computation following [Whi23, Corollary 2.2.13]. $\square$

Definition 4.6.3. For each Taylor–Wiles datum $(\Sigma^{\text{TW}}, \{(T_v, B_v)\}_{v \in \Sigma^{\text{TW}}})$, we obtain another global deformation problem

$$\mathcal{S}(\Sigma^{\text{TW}}) := \left(\overline{\rho}, \varepsilon_\ell^{2-N}, \Sigma^0 \cup \Sigma^{\text{TW}}, \{\mathcal{D}_v^\square\}_{v \in \Sigma \cup \Sigma^{\text{TW}}} \right)$$

where $\mathcal{D}_v^\square$ is the same as in $\mathcal{S}$ for $v \in \Sigma^0$, and for $v \in \Sigma^{\text{TW}}$, $\mathcal{D}_v^\square$ is the local lifting condition of $\overline{\rho}_v$ defined in Proposition 4.6.2. Moreover, for each place $v \in \Sigma^{\text{TW}}$, we adopt the notation from Setup 4.5.1 with subscript

“ v ” added to indicate the place v , and with $(T_\kappa, B_\kappa) = (T_v, B_v)$ and $\bar{g} = \bar{g}_v$. In particular, we have compact subgroups

$$\mathcal{K}_{v,1} \leq \mathcal{K}_{v,0} \leq \mathcal{K}_v = \mathrm{SO}_N(\mathcal{O}_v),$$

such that $\mathcal{K}_{v,1}$ is a normal subgroup of $\mathcal{K}_{v,0}$ with quotient Δ_v .

We then set

- $\Delta_{\Sigma^{\mathrm{TW}}} := \prod_{v \in \Sigma^{\mathrm{TW}}} \Delta_v$;
- $\mathfrak{a}_{\Sigma^{\mathrm{TW}}}$ to be the augmentation ideal of $\mathcal{O}[\Delta_{\Sigma^{\mathrm{TW}}}]$ over $\mathcal{O}$ generated by $\delta - 1$ for all $\delta \in \Delta_{\Sigma^{\mathrm{TW}}}$;
- $\mathfrak{m}_{\Sigma^{\mathrm{TW}}} := \mathfrak{m} \cap \mathbb{T}^{m, \mathfrak{d}, \Sigma \cup \Sigma^{\mathrm{TW}}}$;
- $\mathfrak{n}_i = \prod_{v \in \Sigma_n^{\mathrm{TW}}} \mathfrak{n}_{v,i}$ to be the maximal ideal of $\prod_{v \in \Sigma_n^{\mathrm{TW}}} \mathcal{O}[T^{\mathrm{SO}}(F_v)/(T^{\mathrm{SO}}(\mathcal{O}_v) \cap \mathcal{K}_1)]^{W_{L^{\mathrm{SO}}}}$ (resp. $\prod_{v \in \Sigma_n^{\mathrm{TW}}} \mathcal{O}[T^{\mathrm{SO}}(F_v)/T^{\mathrm{SO}}(\mathcal{O}_v)]^{W_{L^{\mathrm{SO}}}}$) corresponding to $(\bar{g}_v)_{v \in \Sigma_n^{\mathrm{TW}}}$, for each $i \in \{0, 1\}$; and
-

$$\mathcal{K}_i(\Sigma^{\mathrm{TW}}) := \prod_{v \notin \Sigma^{\mathrm{TW}}} \mathcal{K}_v \times \prod_{v \in \Sigma^{\mathrm{TW}}} \mathcal{K}_{v,i}$$

for each $i \in \{0, 1\}$, which are compact open subgroups of $\mathcal{K}$.

In particular, $\mathcal{K}_1(\Sigma^{\mathrm{TW}})$ is a normal subgroup of $\mathcal{K}_0(\Sigma^{\mathrm{TW}})$, and we have a canonical isomorphism

$$(4.1) \quad \mathcal{K}_0(\Sigma^{\mathrm{TW}})/\mathcal{K}_1(\Sigma^{\mathrm{TW}}) \cong \Delta_{\Sigma^{\mathrm{TW}}}.$$

By (4.1), $\mathcal{H}_{\mathcal{K}_1(\Sigma^{\mathrm{TW}})}$ is canonically a module over $\mathcal{O}[\Delta_{\Sigma^{\mathrm{TW}}}]$.

Lemma 4.6.4. *Let the situation be as in Theorem 4.4.2. If $(\Sigma^{\mathrm{TW}}, \{(T_v, B_v)\}_{v \in \Sigma^{\mathrm{TW}}})$ is a Taylor–Wiles datum, then the $\mathcal{O}[\Delta_{\Sigma^{\mathrm{TW}}}]$ -module $\mathcal{H}_{\mathcal{K}_1(\Sigma^{\mathrm{TW}}), \mathfrak{m}_{\Sigma^{\mathrm{TW}}}}$ is finite and free. Moreover, the canonical map*

$$\mathcal{H}_{\mathcal{K}_1(\Sigma^{\mathrm{TW}}), \mathfrak{m}_{\Sigma^{\mathrm{TW}}}}/\mathfrak{a}_{\Sigma^{\mathrm{TW}}} \rightarrow \mathcal{H}_{\mathcal{K}_0(\Sigma^{\mathrm{TW}}), \mathfrak{m}_{\Sigma^{\mathrm{TW}}}}$$

is an isomorphism.

Proof. We modify the proof of [LTX⁺24, Lemma 3.6.5]. For each smooth schemes X over $\mathbb{C}$ and each Abelian group A , we denote by X^{an} the analytification of X , and by $\mathrm{C}_\bullet(X^{\mathrm{an}}, A)$ the complex of singular chains of X^{an} with coefficients in A .

For each $i \in \{0, 1\}$ and each positive integer m , let $\mathcal{K}_i^m(\Sigma^{\mathrm{TW}})$ be the kernel of the composite map

$$\mathcal{K}_i(\Sigma^{\mathrm{TW}}) \rightarrow \mathcal{K} \rightarrow \mathrm{SO}(\Lambda)(\mathcal{O}_F/\ell^m).$$

Then $\mathcal{K}_i^m(\Sigma^{\mathrm{TW}})$ acts trivially on L_ξ/ℓ^m . Hence $\mathcal{L}_\xi \otimes_{\mathcal{O}} \mathcal{O}/\ell^m$ is constant on $\mathrm{Sh}(\mathbf{V}, \mathcal{K}_i^m(\Sigma^{\mathrm{TW}}))$. It follows from (D4) that there is a canonical isomorphism

$$\mathrm{H}_{\mathrm{\acute{e}t}}^{d(\mathbf{V})}(\mathrm{Sh}(\mathbf{V}, \mathcal{K}_i(\Sigma^{\mathrm{TW}})), \mathcal{L}_\xi)_{\mathfrak{m}_{\Sigma^{\mathrm{TW}}}} \cong \mathrm{H}_{\mathrm{\acute{e}t}}^{d(\mathbf{V})}(\mathrm{Sh}(\mathbf{V}, \mathcal{K}_i^m(\Sigma^{\mathrm{TW}})), L_\xi \otimes_{\mathcal{O}} \mathcal{O}/\ell^m)_{\mathfrak{m}_{\Sigma^{\mathrm{TW}}}}^{\mathrm{SO}(\Lambda)(\mathcal{O}_F/\ell^m)}$$

of $\mathcal{O}$ -torsion free modules for every $i \in \{0, 1\}$. By Artin’s comparison theorem between singular cohomologies and étale cohomologies, the dual complex

$$\mathrm{Hom}_{\mathcal{O}}(\mathrm{C}_\bullet(\mathrm{Sh}(\mathbf{V}, \mathcal{K}_i^m(\Sigma^{\mathrm{TW}}))^{\mathrm{an}}, L_\xi \otimes_{\mathcal{O}} \mathcal{O}/\ell^m), \mathcal{O}/\ell^m)$$

calculates $\mathrm{H}_{\mathrm{\acute{e}t}}^\bullet(\mathrm{Sh}(\mathbf{V}, \mathcal{K}_i^m(\Sigma^{\mathrm{TW}})), L_\xi \otimes_{\mathcal{O}} \mathcal{O}/\ell^m)$. It follows from (D4) again that

$$\mathrm{H}_d(\mathrm{Sh}(\mathbf{V}, \mathcal{K}_i^m(\Sigma^{\mathrm{TW}}))^{\mathrm{an}}, L_\xi \otimes_{\mathcal{O}} \mathcal{O}/\ell^m) = 0$$

unless $d = d(\mathbf{V})$.

On the other hand, $\mathcal{K}_i^m(\Sigma^{\mathrm{TW}})$ is neat for every $i \in \{0, 1\}$, which implies that $t^{-1}\mathrm{SO}(\mathbf{V})(F)t \cap \mathcal{K}_i^m(\Sigma^{\mathrm{TW}})$ has no torsion elements for every $t \in \mathrm{SO}(\mathbf{V})(\mathbf{A}_F^\infty)$. Using triangularizations of Shimura varieties as in the proof of [KT17, Lemma 6.9], we see that $\mathrm{C}_\bullet(\mathrm{Sh}(\mathbf{V}, \mathcal{K}_1^m(\Sigma^{\mathrm{TW}}))^{\mathrm{an}}, L_\xi \otimes_{\mathcal{O}} \mathcal{O}/\ell^m)$ is quasiisomorphic to a perfect complex of free $(\mathcal{O}/\ell^m)[\Delta_{\Sigma^{\mathrm{TW}}}]$ -modules; and there is a canonical isomorphism

$$\mathrm{C}_\bullet(\mathrm{Sh}(\mathbf{V}, \mathcal{K}_1^m(\Sigma^{\mathrm{TW}}))^{\mathrm{an}}, L_\xi \otimes_{\mathcal{O}} \mathcal{O}/\ell^m)/\mathfrak{a}_{\Sigma^{\mathrm{TW}}} \cong \mathrm{C}_\bullet(\mathrm{Sh}(\mathbf{V}, \mathcal{K}_0^m(\Sigma^{\mathrm{TW}}))^{\mathrm{an}}, L_\xi \otimes_{\mathcal{O}} \mathcal{O}/\ell^m)$$

of complexes of $(\mathcal{O}/\ell^m)[\mathbb{T}^{m, \mathfrak{d}, \Sigma \cup \Sigma^{\mathrm{TW}}}]$ -modules.

After localizing at $\mathfrak{m}_{\Sigma^{\mathrm{TW}}}$ and taking $\mathrm{SO}(\Lambda)(\mathcal{O}_F/\ell^m)$ -invariants, it follows that the canonical map

$$\mathrm{H}_{\mathrm{\acute{e}t}}^{d(\mathbf{V})}(\mathrm{Sh}(\mathbf{V}, \mathcal{K}_0^m(\Sigma^{\mathrm{TW}})), L_\xi \otimes_{\mathcal{O}} \mathcal{O}/\ell^m)/\mathfrak{a}_{\Sigma^{\mathrm{TW}}} \rightarrow \mathrm{H}_{\mathrm{\acute{e}t}}^{d(\mathbf{V})}(\mathrm{Sh}(\mathbf{V}, \mathcal{K}_1^m(\Sigma^{\mathrm{TW}})), L_\xi \otimes_{\mathcal{O}} \mathcal{O}/\ell^m)$$

is an isomorphism for every positive integer m . Then the lemma follows by passing to the limit for m . $\square$

We now discuss the existence of Taylor–Wiles data, using the notion of G° -adequate subgroups (see Definition 4.2.1).

Lemma 4.6.5. *Let the situation be as in Theorem 4.4.2. Let Ξ be a subset of Σ^0 . For every integer $b \geq \dim_\kappa H^1_{\mathcal{L}^\perp, \Xi}(\text{Gal}_{F, \Sigma}, \text{ad}^0(\bar{\rho})(1))$ (see Equation (3.2)) and every positive integer n , there is a Taylor–Wiles datum $(\Sigma_n^{\text{TW}}, \{(T_v, B_v)\}_{v \in \Sigma_n^{\text{TW}}})$ satisfying*

- (1) $\#\Sigma_n^{\text{TW}} = b$;
- (2) each $v \in \Sigma_n^{\text{TW}}$ is a Taylor–Wiles place of level n ; and
- (3) $R_{\mathcal{S}(\Sigma_n^{\text{TW}})}^{\square_\Xi}$ can be topologically generated over $R_{\mathcal{S}, \Xi}^{\square}$ by

$$\sum_{v \in \Sigma_n^{\text{TW}}} (\dim_\kappa Z(M_{\bar{\rho}(\phi_v)}) - 1) - \sum_{v \in \Xi \cap \Sigma(\ell)} [F_v : \mathbb{Q}_\ell] (\dim G_\kappa - \dim B_\kappa)$$

elements.

Proof. We modify the proof of [Tho12, Theorem 4.4]. By Proposition 3.3.9, Proposition 3.5.11, and Proposition 3.6.3, for each place $v \in \Sigma$,

$$\dim_\kappa L(\mathcal{D}_v) - \dim_\kappa H^0(F_v, \text{ad}^0(\bar{\rho})) = \begin{cases} [F_v : \mathbb{Q}_\ell] (\dim G_\kappa - \dim B_\kappa) & \text{if } v \in \Sigma(\ell) \\ 0 & \text{if } v \notin \Sigma(\ell) \end{cases}.$$

For any Taylor–Wiles datum $(\Sigma^{\text{TW}}, \{(T_v, B_v)\}_{v \in \Sigma^{\text{TW}}})$, it follows from the Wiles’ formula (see Lemma 3.2.7(5)), 4.6.2, and the proof of [CHT08, Corollary 2.2.12] that $R_{\mathcal{S}(\Sigma^{\text{TW}})}^{\square_\Xi}$ can be generated over $R_{\mathcal{S}, \Xi}^{\square}$ by

$$(4.2) \quad \begin{aligned} \dim_\kappa H^1_{\mathcal{S}(\Sigma^{\text{TW}}), \Xi}(\text{Gal}_{F, \Sigma}, \text{ad}^0(\bar{\rho})) &= \dim_\kappa H^1_{\mathcal{L}(\Sigma^{\text{TW}})^\perp, \Xi}(\text{Gal}_{F, \Sigma}, \text{ad}^0(\bar{\rho})(1)) + \sum_{v \in \Sigma^{\text{TW}}} (\dim_\kappa Z(M_{\bar{\rho}(\phi_v)}) - 1) \\ &\quad - \sum_{v \in \Xi \cap \Sigma(\ell)} [F_v : \mathbb{Q}_\ell] (\dim G_\kappa - \dim B_\kappa) - \dim_\kappa H^0(\text{Gal}_{F, \Sigma}, \text{ad}^0(\bar{\rho})(1)) \end{aligned}$$

elements. It follows from [Whi23, Proposition 2.2.18 and Lemma 2.2.20] that there exists a set Σ_n^{TW} of Taylor–Wiles places satisfying Conditions (1)–(2), such that

$$\dim_\kappa H^1_{\mathcal{L}(\Sigma_n^{\text{TW}})^\perp, \Xi}(\text{Gal}_{F, \Sigma}, \text{ad}^0(\bar{\rho})(1)) = 0.$$

Thus the lemma follows. $\square$

Corollary 4.6.6. *Let the situation be as in Theorem 4.4.2. Let Ξ be a subset of Σ^0 . Then there exists a positive integers $t, b \in \mathbb{Z}_+$, such that for every positive integer n , there is a Taylor–Wiles datum $(\Sigma_n^{\text{TW}}, \{(T_v, B_v)\}_{v \in \Sigma_n^{\text{TW}}})$ satisfying*

- (1) $\#\Sigma_n^{\text{TW}} = b$;
- (2) $\|v\| \equiv 1 \pmod{\ell^n}$ for each $v \in \Sigma_n^{\text{TW}}$.
- (3) $R_{\mathcal{S}(\Sigma_n^{\text{TW}})}^{\square_\Xi}$ can be topologically generated over $R_{\mathcal{S}, \Xi}^{\square}$ by

$$g_{t, \Xi} := t - \sum_{v \in \Xi \cap \Sigma(\ell)} [F_v : \mathbb{Q}_\ell] (\dim G_\kappa - \dim B_\kappa)$$

elements.

- (4) $\Delta_{\Sigma_n^{\text{TW}}}$ is an Abelian ℓ -group with t cyclic factors.

Proof. This follows from Lemma 4.6.5 and the Pigeonhole principle. $\square$

We now prove our main result.

Proof of Theorem 4.4.2. We fix a universal lifting

$$\rho_{\mathcal{S}} : \text{Gal}_F \rightarrow G(R_{\mathcal{S}}).$$

Let Ξ be a fixed subset of Σ^0 . We fix positive integer $b, t \in \mathbb{Z}_+$ and a Taylor–Wiles datum $(\Sigma_n^{\text{TW}}, \{(T_v, B_v)\}_{v \in \Sigma_n^{\text{TW}}})$ of size b for each positive integer n from Lemma 4.6.6. For each $v \in \Sigma_n^{\text{TW}}$, we

- let $\text{Art}_v : F_v^\times \xrightarrow{\sim} \text{Gal}_{F_v}^{\text{ab}}$ be the local Artin map; and
 - let $\varpi_v \in F_v$ be a uniformizer, such that $\text{Art}_v(\varpi_v)$ coincides with the image of ϕ_v^{-1} in $\text{Gal}_{F_v}^{\text{ab}}$.
- We adopt notation from Definition 4.6.3, and
- set $\mathcal{H} := \mathcal{H}_{\mathcal{K}_0(\emptyset), \mathfrak{m}}$;
 - let $\mathcal{H}_{i,n}$ be the localized module $\left(\mathcal{H}_{\mathcal{K}_i(\Sigma_n^{\text{TW}}), \mathfrak{m}_{\Sigma_n^{\text{TW}}}} \right)_{\mathfrak{n}_i}$ for each $n \in \mathbb{Z}_+$ and $i \in \{0, 1\}$; and
 - let $\mathbb{T}_{i,n}$ be the image of $\mathbb{T}^{m, \mathfrak{d}, \Sigma_n^{\text{TW}}}$ in $\text{End}_{\mathcal{O}}(\mathcal{H}_{i,n})$, for each $n \in \mathbb{Z}_+$ and $i \in \{0, 1\}$.

For each $n \in \mathbb{Z}_+$, since the canonical map $\mathcal{H} \rightarrow \mathcal{H}_{\mathcal{K}, \mathfrak{m}_{\Sigma_n^{\text{TW}}}}$ is an isomorphism, we obtain canonical surjections

$$(4.4) \quad \mathbb{T}_{1,n} \twoheadrightarrow \mathbb{T}_{0,n} \twoheadrightarrow \mathbb{T}_{\mathfrak{m}}$$

of rings over $\mathcal{O}$. Similar to the construction of $\rho_{\mathfrak{m}}$, we obtain liftings $\rho_{i,n}$ of $\bar{\rho}$ to $\mathbb{T}_{i,n}$ that are compatible with the maps in (4.4).

Analogous to [Tho12, Propositions 5.9, 5.12], we claim that

- (1) $\rho_{1,n}$ satisfies the global deformation problem $\mathcal{S}(\Sigma_n^{\text{TW}})$ (see Definition 4.6.3). Moreover, for each $v \in \Sigma_n^{\text{TW}}$, the induced map $\Delta_v \rightarrow \mathbb{T}_{1,n}$ as defined in [Whi23, pp.59–60] coincides with the scalar action of Δ_v on $\mathcal{H}_{1,n}$ arising from viewing $\Delta_v = T^{\text{SO}}(\mathcal{O}_v)/(T^{\text{SO}}(\mathcal{O}_v) \cap \mathcal{K}_{v,1})$ as Hecke operators.
- (2) The natural map

$$\mathcal{H}_{0,n} \rightarrow \mathcal{H}$$

is an isomorphism for every $n \in \mathbb{Z}_+$. In particular, the canonical homomorphism $\mathbb{T}_{0,n} \rightarrow \mathbb{T}_{\mathfrak{m}}$ is an isomorphism, and $\rho_{0,n}$ are equivalent to $\rho_{\mathfrak{m}}$ as liftings of $\bar{\rho}$.

The first claim follows from applying [Whi23, Proposition 2.5.3] to the Galois representation associated to the maximal ideals of $\mathbb{T}_{1,n}[1/\ell]$, by the construction of $\rho_{1,n}$. Here we note (2.2) and Corollary 4.5.4 together guarantee that the hypotheses of [Whi23, Proposition 2.5.3] hold, except for the fact that the local Galois representations are only valued in G° rather than G' , which does not affect the proof. The second claim follows from applying [Whi23, Corollary 2.4.14] to the module $\mathcal{H}_{\mathcal{K}_{\text{Iw}}(\Sigma_n^{\text{TW}}), \mathfrak{m}_{\Sigma_n^{\text{TW}}}}$, where $\mathcal{K}_{\text{Iw}}(\Sigma_n^{\text{TW}})$ is defined such that $\mathcal{K}_{\text{Iw}}(\Sigma_n^{\text{TW}})_v = \text{Iw}_v$ for $v \in \Sigma_n^{\text{TW}}$ and $\mathcal{K}_{\text{Iw}}(\Sigma_n^{\text{TW}})_v = \mathcal{K}_v$ for $v \notin \Sigma_n^{\text{TW}}$. Here $\text{Iw}_v \subset \text{SO}(\mathcal{O}_v)$ is the standard Iwahori subgroup corresponding to B^{SO} .

It follows from (1) above that $\rho_{1,n}$ satisfies the global deformation problem $\mathcal{S}(\Sigma_n^{\text{TW}})$, which induces a canonical surjective homomorphism

$$\varphi_n : R_{\mathcal{S}(\Sigma_n^{\text{TW}})} \rightarrow \mathbb{T}_{1,n}.$$

of $\mathcal{O}[\Delta_{\Sigma_n^{\text{TW}}}]$ -modules.

By (2) and Lemma 4.6.4, we obtain a canonical commutative diagram

$$\begin{array}{ccc} R_{\mathcal{S}(\Sigma_n^{\text{TW}})}/\mathfrak{a}_{\Sigma_n^{\text{TW}}} & \xrightarrow{\sim} & R_{\mathcal{S}} \\ \downarrow \varphi_n/\mathfrak{a}_{\Sigma_n^{\text{TW}}} & & \downarrow \varphi \\ \mathbb{T}_{1,n}/\mathfrak{a}_{\Sigma_n^{\text{TW}}} & \xrightarrow{\sim} & \mathbb{T}_{\mathfrak{m}} \end{array}$$

of ring over $\mathcal{O}$, where the horizontal maps are isomorphisms.

For each $n \in \mathbb{Z}_+$, we choose a universal lifting

$$\rho_{\mathcal{S}(\Sigma_n^{\text{TW}})} : \text{Gal}_F \rightarrow G(R_{\mathcal{S}(\Sigma_n^{\text{TW}})})$$

of $\rho_{\mathcal{S}}$ to $R_{\mathcal{S}(\Sigma_n^{\text{TW}})}$. By Proposition 3.2.8(3), we obtain isomorphisms

$$R_{\mathcal{S}} \hat{\otimes} \mathcal{T}_{\Xi} \cong R_{\mathcal{S}}^{\square_{\Xi}}, \quad R_{\mathcal{S}(\Sigma^{\text{TW}})} \hat{\otimes} \mathcal{T}_{\Xi} \cong R_{\mathcal{S}(\Sigma^{\text{TW}})}^{\square_{\Xi}}$$

of rings over $\mathcal{O}$. In particular, we can choose a surjective local homomorphism $R_{\mathcal{S}}^{\square_{\Xi}} \rightarrow R_{\mathcal{S}}$, making $R_{\mathcal{S}}$ an algebra over $R_{\mathcal{S}, \Xi}^{\square}$.

We set

$$S_\infty := \mathcal{T}_\Xi[[Y_1, \dots, Y_t]],$$

where $\mathcal{T}_\Xi$ is the coordinate ring of the formal smooth group $\prod_{v \in \Xi} (G')^\wedge$ over $\mathcal{O}$. Let $\mathfrak{a}_\infty \subset S_\infty$ be the augmentation ideal over $\mathcal{O}$. Set

$$R_\infty := R_{\mathcal{S}, \Xi}^\square[[Z_1, \dots, Z_{g_{t, \Xi}}]],$$

where $g_{t, \Xi}$ is the integer appearing in Lemma 4.6.6. Applying the usual patching lemma (see the proof of [BLGG11, Thm 3.6.1] or [Tho12, Lemma 6.10]), it follows that

- there exists a local homomorphism $S_\infty \rightarrow R_\infty$ over $\mathcal{O}$ such that we have a surjective homomorphism $R_\infty/\mathfrak{a}_\infty R_\infty \rightarrow R_{\mathcal{S}}$ of rings over $R_{\mathcal{S}, \Xi}^\square$.
- there exists a R_∞ -module $\mathcal{H}_\infty$ such that
 - $\mathcal{H}_\infty$ is finite free over S_∞ , and
 - $\mathcal{H}_\infty/\mathfrak{a}_\infty \mathcal{H}_\infty$ is isomorphic to $\mathcal{H}$ as $R_\infty/\mathfrak{a}_\infty R_\infty$ -modules.

In particular, we have

$$\text{depth}_{R_\infty} \mathcal{H}_\infty \geq \text{depth}(S_\infty) = 1 + \dim G'_\kappa \cdot \#\Xi + t.$$

On the other hand, it follows from Proposition 3.3.9, Proposition 3.5.11, and Proposition 3.6.3 that $R_{\mathcal{S}, \Xi}^\square$ is a formal power series over $\mathcal{O}$ in

$$\dim G'_\kappa \cdot \#\Xi + \sum_{v \in \Xi \cap \Sigma(\ell)} [F_v : \mathbb{Q}_\ell] (\dim G_\kappa - \dim B_\kappa)$$

variables. It follows that R_∞ is a regular local ring of dimension

$$1 + \dim G'_\kappa \cdot \#\Xi + \sum_{v \in \Xi \cap \Sigma(\ell)} [F_v : \mathbb{Q}_\ell] (\dim G_\kappa - \dim B_\kappa) + g_{t, \Xi} = 1 + \dim G'_\kappa \cdot \#\Xi + t.$$

Thus it follows from the Auslander–Buchsbaum theorem that $\mathcal{H}_\infty$ is a finite free R_∞ -module.

Then the R_∞ -module $\mathcal{H} = \mathcal{H}_\infty/\mathfrak{a}_\infty \mathcal{H}_\infty$ is finite free over $R_\infty/\mathfrak{a}_\infty R_\infty$. But the action of $R_\infty/\mathfrak{a}_\infty R_\infty$ on $\mathcal{H}$ factors through the surjective homomorphism $R_\infty/\mathfrak{a}_\infty R_\infty \rightarrow R_{\mathcal{S}}$, implying that this surjection must be an isomorphism. Thus $\mathcal{H}$ is a finite free $R_{\mathcal{S}}$ -module. As a result, the map $\varphi : R_{\mathcal{S}} \rightarrow \mathbb{T}_m$ is an isomorphism. $\square$

5. RIGIDITY

5.1. Rigidity of odd symmetric powers of elliptic curves. In this subsection, we study rigidity of reductions of odd symmetric powers of Galois representation attached to elliptic curves.

Let A be an elliptic curve over F . For every rational prime ℓ , we fix an isomorphism $H_{\text{ét}}^1(A_{\overline{F}}, \mathbb{Z}_\ell) \cong \mathbb{Z}_\ell^{\oplus 2}$, and let

$$\rho_{A, \ell} : \Gamma_F \rightarrow \text{GL}(H_{\text{ét}}^1(A_{\overline{F}}, \mathbb{Z}_\ell)) \cong \text{GL}_2(\mathbb{Z}_\ell)$$

be the continuous homomorphism attached to A . Then we obtain a continuous homomorphism

$$\rho_{A, m, \ell} := \text{Sym}^{2m-1}(\rho_{A, \ell}) : \text{Gal}_F \rightarrow \text{GSp}_{2m}(\mathbb{Z}_\ell).$$

Recall that GSp_{2m} is defined by the standard alternating pairing on $\mathbb{Z}_\ell^{\oplus (2m)}$ given by the matrix J'_{2m} ; see Setup 3.0.1.

When v is a finite place of F and ℓ is a rational prime, let G be the smooth group scheme GSp_{2m} over $\mathbb{Z}_\ell$, and we work in Setup 3.2.1 with $\Gamma = \text{Gal}_{F_v}$ and $\chi = \varepsilon_\ell^{1-2m}$.

Proposition 5.1.1. *Let v be a finite place of F . For all but finitely many rational primes $\ell > 2m$, every lifting of $\bar{\rho}_{A, m, \ell}|_{\text{Gal}_{F_v}}$ to an object R of $\mathcal{C}_{\mathbb{Z}_\ell}$ (with respect to $\chi = \varepsilon_\ell^{1-2m}$) is minimally ramified in the sense of Definition 3.5.8.*

Proof. Choose a finite totally ramified extension F'_v of F_v inside $\overline{F_v}$ such that $A' := A \otimes_F F'_v$ has either good or multiplicative reduction. We adopt the notation of §3.5, and let T'_v be the image of the subgroup $\text{Gal}(\overline{F_v}/F'_v)$ in $T_v := \text{Gal}_{F_v}/P_v$. We fix an isomorphism between T_v and the $\|v\|$ -tame group $T_{\|v\|}$. Assume that $\ell \nmid [F'_v : F_v]$, so that $T'_v = T_v$. Choose a finite unramified extension E of $\mathbb{Q}_\ell$ contained in $\overline{\mathbb{Q}_\ell}$ with ring of integers $\mathcal{O}$ and residue field κ that satisfies Assumption 3.5.2 for $\bar{\rho}$. Let $\mathfrak{T} = \mathfrak{T}(\bar{\rho}_{A, m, \ell, v} \otimes \kappa)$ be the set of isomorphism classes of absolutely irreducible representations of P_v appearing in $\text{Std} \circ \bar{\rho}_{A, m, \ell, v} \otimes \kappa$.

We first consider the case when A' has good reduction. Let $\alpha, \beta \in \overline{\mathbb{Q}_\ell}$ be the two eigenvalues of $\rho_{A', \ell}(\phi_q)$. Then α and β are Weil $\|v\|^{-1/2}$ -numbers in $\overline{\mathbb{Q}}$, which depends only on A' , not on ℓ . We further assume that ℓ satisfies $\alpha, \beta \in (\overline{\mathbb{Z}_\ell})^\times$, and that the image of the set

$$\{(\alpha/\beta)^{2m}, (\alpha/\beta)^{2m-1}, \dots, (\alpha/\beta)^{-2m}\}$$

in $\overline{\mathbb{F}_\ell}^\times$ does not contain $\|v\|$. It follows that for any $\tau \in \mathfrak{T}$, every lifting of $W_\tau(\rho_{A, m, \ell, v} \otimes \kappa)$ is actually unramified by Lemma 3.4.7(2) if we assume that $\ell \nmid \prod_{i=1}^{2m+1} (\|v\|^i - 1)$.

We then consider the case when A' has multiplicative reduction. Let $u = \text{ord}_v(j(A)) < 0$ be the valuation of the j -invariant of A . Assume further that ℓ is coprime to u . Then $\rho_{A, \ell}(t)$ is conjugate to $\pm(1 + N_2)$ in $\text{GSp}_2(\mathbb{Z}_\ell)$, which implies that $\text{Sym}^{2m-1} \rho_{A, \ell}(t)$ is conjugate to $\pm \exp(N_{2m})$ in $\text{GSp}_{2m}(\mathbb{Z}_\ell)$. It follows that $\mathfrak{T}$ is a singleton. Let τ be the unique element of $\mathfrak{T}$, then every lifting of $W_\tau(\bar{\rho}_{A, m, \ell, v} \otimes \kappa)$ is minimally unramified because N_{2m} is of maximal rank. $\square$

Combining this proposition with Proposition 2.2.3, we have the following immediate corollary.

Corollary 5.1.2. *Let Σ be a finite set of finite places of F containing Σ^{bad} such that A has good reduction outside Σ . Then all for but finitely many rational primes ℓ , $\bar{\rho}_{A, m, \ell}$ is rigid for $(\Sigma, \emptyset)$ in the sense of Definition 4.1.1 (with $\mathcal{O} = \mathbb{Z}_\ell$).*

5.2. Rigidity of automorphic Galois representations. In this subsection, we study rigidity of reductions of automorphic Galois representations. We work in the following setting.

Setup 5.2.1.

- Let Π be a $\mathfrak{d}$ -REASDC representation of $\text{GL}_{2m}(\mathbf{A}_F)$ (see Definition 2.2.2).
- Let Σ^Π be the smallest (finite) set of finite places of F containing Σ^{bad} such that Π_v is unramified for every finite place v of F not in Σ^Π .
- Let $E \subset \mathbb{C}$ be a strong coefficient field of Π (see Definition 2.2.5).

For each finite place λ of E , we obtain a continuous homomorphism $\rho_{\Pi, \lambda} : \text{Gal}_F \rightarrow \text{GL}_{2m}(E_\lambda)$.

Conjecture 5.2.2. Fix a finite set Σ of finite places of F containing Σ^Π . Suppose that the base change of Π to $F(\sqrt{\mathfrak{d}})$ is also cuspidal. Then for all but finitely many primes λ of E with underlying rational prime ℓ ,

- (1) $\bar{\rho}_{\Pi, \lambda}|_{\text{Gal}(\bar{F}/F(\mu_\ell(\sqrt{\mathfrak{d}}))}$ is absolutely irreducible; and
- (2) $\bar{\rho}_{\Pi, \lambda}$ is rigid for $(\Sigma, \emptyset)$ in the sense of Definition 4.1.1.

Concerning Conjecture 5.2.2, we have the following theorem.

Theorem 5.2.3. *Suppose there exists a finite place v_0 of F split in $F(\sqrt{\mathfrak{d}})$ such that Π_{v_0} is supercuspidal, then Conjecture 5.2.2 holds.*

Proof. For part (1): We denote $F(\sqrt{\mathfrak{d}})$ by $\dot{F}$, and choose a place w_0 of $\dot{F}$ over v_0 . In particular, we have a natural isomorphism $F_{v_0} \cong \dot{F}_{w_0}$. Since Π_{v_0} is supercuspidal, the continuous representation $\rho_{\Pi_{v_0}} : W_{F_{v_0}} \rightarrow \text{GL}_{2m}(\mathbb{C})$ associated with Π_{v_0} via the local Langlands correspondence is irreducible. By Clifford's theorem, there exists an irreducible representation τ of $I_{F_{v_0}}$ (with complex coefficients) and a character χ of $I_{F_{v_0}} \rtimes \langle \varphi^b \rangle$, such that $\rho_{\Pi_{v_0}}$ is isomorphic to $\text{Ind}_{I_{F_{v_0}} \rtimes \langle \varphi^b \rangle}^{W_{F_{v_0}}}(\tau \otimes \chi)$, where b is the smallest positive integer satisfying $\tau^{\phi_{v_0}^b} \cong \tau$. We may choose a finite extension E' of E inside $\mathbb{C}$ and a finite set Λ' of finite places of E' , such that both τ and χ are defined over $\mathcal{O}_{E', (\Lambda')}$. In particular, we may assume that the image of $\rho_{\Pi_{v_0}}$ is contained in $\text{GL}_{2m}(\mathcal{O}_{E', (\Lambda')})$.

Let Λ'_1 be the smallest set of finite places of E' containing Λ' such that every finite place λ' of E' not in Λ'_1 satisfies

- $\bar{\tau}_{\lambda'} := \tau \otimes_{\mathcal{O}_{E', (\Lambda')}} \bar{\kappa}_{\lambda'}$ is irreducible; and
- b is the smallest positive integer k that satisfies $\bar{\tau}_{\lambda'}^{\phi_{w_0}^k}$ is conjugate to $\bar{\tau}_{\lambda'}$.

Let Λ_1 be the finite set of finite places of E underlying Λ'_1 , and let Λ_2 be the union of Λ_1 and the set of finite places of E whose underlying rational prime ℓ satisfies either $\ell \leq 2m(b_\xi - a_\xi) + 1$ or $\Sigma(\ell) \cap \Sigma^\Pi \neq \emptyset$. Take a finite place λ not in Λ_2 with underlying rational prime ℓ . Then $\bar{\rho}_{\Pi, \lambda}|_{\text{Gal}(\bar{F}/\dot{F})}$ is absolutely

irreducible by Proposition 2.2.3(2). By abuse of notation, we regard $\bar{\rho}_{\Pi,\lambda}|_{\text{Gal}(\bar{F}/\dot{F})}$ as a representation with $\overline{\kappa\lambda}$ -coefficients. Since $[\dot{F}(\mu_\ell) : \dot{F}]$ is coprime to ℓ , the representation $\bar{\rho}_{\Pi,\lambda}|_{\text{Gal}(\bar{F}/\dot{F}(\mu_\ell))}$ is semisimple. We claim that $\bar{\rho}_{\Pi,\lambda}|_{\text{Gal}(\bar{F}/\dot{F})}$ is an induction of an irreducible representation ρ' of $\text{Gal}(\bar{F}/F')$ for some field extension $\dot{F} \subset F' \subset \dot{F}(\mu_\ell)$ such that $[F' : \dot{F}]$ equals the number of irreducible summands of $\bar{\rho}_{\Pi,\lambda}|_{\text{Gal}(\bar{F}/\dot{F}(\mu_\ell))}$. By [CG13, Lemma 4.3], it suffices to show that the irreducible summands of $\bar{\rho}_{\Pi,\lambda}|_{\text{Gal}(\bar{F}/\dot{F}(\mu_\ell))}$ are pairwise nonisomorphic. Since v is unramified in $\dot{F}(\mu_\ell)$, it suffices to check that the irreducible summands of $\bar{\rho}_{\Pi,\lambda}|_{I_{F_v}}$ are pairwise non-isomorphic. But this follows from the definition of Λ'_1 .

By our definition of Λ_2 and Proposition 2.2.3(3), $\bar{\rho}_{\Pi,\lambda}|_{\text{Gal}(\bar{F}/\dot{F})}$ is crystalline with regular Fontaine–Laffaille weights in $[a_\xi, b_\xi]$ and $\ell > 2m(b_\xi - a_\xi) + 1 \geq (b_\xi - a_\xi) + 2$. Thus we must have $F' = \dot{F}$ by [LTX⁺24, Lemma 4.7], and then (1) follows.

For part (2): Let ξ be the Archimedean weights of Π (see Definition 2.2.2). Let Λ_2 be the finite set of finite places of E from part (1). We show that each of the four conditions in Definition 4.1.1 excludes only finitely many finite places λ of E . Condition (2) is empty. Condition (3) holds if the underlying rational prime ℓ of λ satisfies $\ell \geq 2(b_\xi - a_\xi) + 2$ and $\Sigma(\ell) \cap \Sigma = \emptyset$, by Proposition 2.2.3(3). Condition (4) is automatic by Proposition 2.2.3(2). It remains to consider Condition (1).

Let λ be a finite place of E whose underlying rational prime ℓ satisfies

$$\Sigma(\ell) \cap \Sigma = \emptyset, \quad \ell > 2(2m + 1), \quad \ell > 2(b_\xi - a_\xi) + 2,$$

and $\ell > \|v\|^{2m}$ for every place v in Σ . Then

- (a) ℓ is unramified in F ;
- (b) Π_v is unramified for every $v \in \Sigma(\ell)$;
- (c) $\text{Std} \circ \bar{\rho}_{\Pi,\lambda}|_{\text{Gal}(\bar{F}/F(\mu_\ell)(\sqrt{\mathfrak{d}}))}$ is absolutely irreducible; which implies $\bar{\rho}_{\Pi,\lambda}(\text{Gal}(\bar{F}/F(\mu_\ell)))$ is G° -adequate by Proposition 4.2.2;
- (d) Proposition 3.3.9 holds for the local deformation problem $\mathcal{D}^{\text{FL}}$ of $\bar{\rho}_{\Pi,\lambda,v}$ (see Definition 3.3.8) for every $v \in \Sigma(\ell)$; and
- (e) Proposition 3.5.11 holds for the local deformation problem $\mathcal{D}^{\text{min}}$ of $\bar{\rho}_{\Pi,\lambda,v}$ (see Definition 3.5.10) for every $v \in \Sigma$.

We work in the setting of Setup 3.0.1 with $E = E_\lambda$ plus setting of Setup 3.2.1 with $(\Gamma, \bar{\rho}, \chi) = (\text{Gal}_F, \bar{\rho}, \varepsilon_\ell^{2-N})$, and fix an isomorphism $\iota_\ell : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}_\ell}$. For any collection $\mathcal{D}_\Sigma^\square := \{\mathcal{D}_v^\square | v \in \Sigma\}$ in which $\mathcal{D}_v^\square$ is an irreducible component of $\text{Spf } R_{\bar{\rho}_{\Pi,\lambda,v}}^\square$ for every $v \in \Sigma$, we define a global deformation problem (see Definition 3.2.5)

$$\mathcal{S}(\mathcal{D}_\Sigma^\square) := \left(\bar{\rho}_{\Pi,\lambda}, \varepsilon_\ell^{2-N}, \Sigma \cup \Sigma(\ell), \{\mathcal{D}_v^\square\}_{v \in \Sigma \cup \Sigma(\ell)} \right),$$

where for each $v \in \Sigma$, $\mathcal{D}_v^\square$ is the prescribed irreducible component (which is a local deformation problem by Proposition 3.5.11(2)); and for each $v \in \Sigma(\ell)$, $\mathcal{D}_v^\square$ is the local deformation $\mathcal{D}^{\text{FL},\square}$ of $\bar{\rho}_{\Pi,\lambda}$ from Definition 3.3.8.

It follows from Theorem 4.3.2 that $R_{\mathcal{S}(\mathcal{D}_\Sigma^\square)}$ is a finite $\mathcal{O}$ -module. On the other hand, it follows from Proposition 3.2.8 and the proof of [CHT08, Corollary 2.3.5] that

$$\begin{aligned} \dim R_{\mathcal{S}(\mathcal{D}_\Sigma^\square)} &\geq 1 + \sum_{v \in \Sigma \cup \Sigma(\ell)} (\dim_\kappa L(\mathcal{D}_v) - \dim_\kappa H^0(\text{Gal}_{F_v}, \text{ad}^0(\bar{\rho}))) - \dim_\kappa H^0(G_{F, \Sigma \cup \Sigma(\ell)}, \text{ad}^0(\bar{\rho})(1)) \\ &\quad - [F : \mathbb{Q}](\dim G_\kappa - \dim B_\kappa). \end{aligned}$$

It follows from (d)–(e) that for each place $v \in \Sigma$,

$$\dim_\kappa L(\mathcal{D}_v) - \dim_\kappa H^0(\text{Gal}_{F_v}, \text{ad}^0(\bar{\rho})(1)) = \begin{cases} [F_v : \mathbb{Q}_\ell](\dim G_\kappa - \dim B_\kappa) & \text{if } v|\ell \\ 0 & \text{if } v \nmid \ell \end{cases}.$$

The assumption that $\bar{\rho}_{\Pi,\lambda}(\text{Gal}(\bar{F}/F(\mu_\ell)))$ is adequate implies that $H^0(G_{F, \Sigma \cup \Sigma(\ell)}, \text{ad}^0(\bar{\rho})(1)) = 0$ by Definition 4.2.1. Thus we have $\dim R_{\mathcal{S}(\mathcal{D}_\Sigma^\square)} \geq 1$. In particular, $\text{Spec } R_{\mathcal{S}(\mathcal{D}_\Sigma^\square)}[\frac{1}{\ell}]$ is nonzero.

Choose an arbitrary $\overline{\mathbb{Q}_\ell}$ -point of $\text{Spec } R_{\mathcal{S}(\mathcal{D}_\Sigma^\square)}[\frac{1}{\ell}]$. Via ι_ℓ^{-1} and classical modularity theorems (see, for example, [Gue11, Theorem 4.2] strengthened using Thorne’s adequacy condition [Tho12]), we obtain a $\mathfrak{d}$ -REASDC representation $\Pi(\mathcal{D}_\Sigma^\square)$ satisfying

- $\Pi(\mathcal{D}_\Sigma^\square)$ is unramified outside $\Sigma \cup \Sigma^\infty$;
- for each place v of F in Σ , there is an open compact subgroup U_v of $\mathrm{GL}_{2m}(F_v)$ depending only on Π_v , such that $\Pi(\mathcal{D}_\Sigma^\square)_v$ has nontrivial vectors fixed by U_v ;
- the Archimedean weights of $\Pi(\mathcal{D}_\Sigma^\square)$ equals ξ ;
- $\rho_{\Pi(\mathcal{D}_\Sigma^\square), \iota_\ell}$ and $\rho_{\Pi, \lambda} \otimes_{E_\lambda} \overline{\mathbb{Q}_\ell}$ are residually isomorphic.

Note that the second property is a consequence of Proposition 2.2.3(2), Corollary 3.5.7 (which is applicable since $\ell > \|v\|^{2m}$), and the fact that irreducible admissible representations in a common Bernstein component have a common level. Note that there are only finitely many isomorphism classes of $\mathfrak{d}$ -REASDC representation of $\mathrm{GL}_{2m}(\mathbf{A}_F)$ satisfying the first three properties. By the strong multiplicity one property of GL_{2m} [PS79], when ℓ is large, Π is the only $\mathfrak{d}$ -REASDC representation of $\mathrm{GL}_{2m}(\mathbf{A}_F)$ up to isomorphism satisfying all the four properties.

By Proposition 3.5.11, the theorem would follow if we show that for any two different collections $\mathcal{D}_\Sigma^\square$ and $\mathcal{D}_\Sigma^{\square, '}$, the $\mathfrak{d}$ -REASDC representations $\Pi(\mathcal{D}_\Sigma^\square)$ and $\Pi(\mathcal{D}_\Sigma^{\square, '})$ are not isomorphic. To show this, let v be a place in Σ , and let x be the $\overline{\mathbb{Q}_\ell}$ -point of $\mathrm{Spec} \mathcal{R}_{\rho_{\Pi, \lambda, v}}^\square[\frac{1}{\ell}]$ corresponding to $\rho_{\Pi(\mathcal{D}_\Sigma^\square), \iota_\ell}|_{F_v}$. The dimension of the tangent space of $\mathrm{Spec} \mathcal{R}_{\rho_{\Pi, \lambda, v}}^\square[\frac{1}{\ell}]$ at x is equal to

$$\begin{aligned} Z^1(F_v, \mathrm{ad}^0(\rho)_{\Pi(\mathcal{D}_\Sigma^\square), \iota_\ell}|_{F_v}) &= \dim G'_\kappa + \dim_{\overline{\mathbb{Q}_\ell}} H^1(F_v, \mathrm{ad}^0(\rho)_{\Pi(\mathcal{D}_\Sigma^\square), \iota_\ell}|_{F_v}) - H^0(F_v, \mathrm{ad}^0(\rho)_{\Pi(\mathcal{D}_\Sigma^\square), \iota_\ell}|_{F_v}) \\ &= \dim G'_\kappa + \dim_{\overline{\mathbb{Q}_\ell}} H^2(F_v, \mathrm{ad}^0(\rho)_{\Pi(\mathcal{D}_\Sigma^\square), \iota_\ell}|_{F_v}) \\ &= \dim G'_\kappa + \dim_{\overline{\mathbb{Q}_\ell}} H^0(F_v, \mathrm{ad}^0(\rho)_{\Pi(\mathcal{D}_\Sigma^\square), \iota_\ell}|_{F_v}(1)). \end{aligned}$$

Since $\Pi(\mathcal{D}_\Sigma^\square)_v$ is tempered by Proposition 2.2.3(1), we have $\dim_{\overline{\mathbb{Q}_\ell}} H^0(F_v, \mathrm{ad}^0(\rho)_{\Pi(\mathcal{D}_\Sigma^\square), \iota_\ell}|_{F_v}(1)) = 0$ by the proof of [BLGGT14, Lemma 1.3.2(1)]. It follows from Proposition 3.5.11(1) that $\mathrm{Spec} \mathcal{R}_{\rho_{\Pi, \lambda, v}}^\square[\frac{1}{\ell}]$ is smooth at x . Thus x cannot lie on two irreducible components. Considering each place v in Σ , we conclude that $\Pi(\mathcal{D}_\Sigma^\square)$ and $\Pi(\mathcal{D}_\Sigma^{\square, '})$ are not isomorphic.

The theorem is proved. $\square$

REFERENCES

- [ABPS17] A.-M. Aubert, P. Baum, R. Plymen, and M. Solleveld, *The principal series of p -adic groups with disconnected center*, Proc. Lond. Math. Soc. (3) **114** (2017), no. 5, 798–854. MR3653247
- [AG17] H. Atobe and W. T. Gan, *On the local Langlands correspondence and Arthur conjecture for even orthogonal groups*, Represent. Theory **21** (2017), 354–415. MR3708200
- [AMS24] A.-M. Aubert, A. Moussaoui, and M. Solleveld, *Affine hecke algebras for Langlands parameters* (2024), available at [arXiv:1701.03593](https://arxiv.org/abs/1701.03593).
- [Art13] J. Arthur, *The endoscopic classification of representations*, American Mathematical Society Colloquium Publications, vol. 61, American Mathematical Society, Providence, RI, 2013. Orthogonal and symplectic groups. MR3135650
- [Bal12] S. Balaji, *G -valued potentially semi-stable deformation rings*, ProQuest LLC, Ann Arbor, MI, 2012. Thesis (Ph.D.)—The University of Chicago. MR3152673
- [BG19] R. Bellovin and T. Gee, *G -valued local deformation rings and global lifts*, Algebra Number Theory **13** (2019), no. 2, 333–378. MR3927049
- [BK90] S. Bloch and K. Kato, *L -functions and Tamagawa numbers of motives*, The Grothendieck Festschrift, Vol. I, 1990, pp. 333–400. MR1086888
- [BLGG11] T. Barnet-Lamb, T. Gee, and D. Geraghty, *The Sato-Tate conjecture for Hilbert modular forms*, J. Amer. Math. Soc. **24** (2011), no. 2, 411–469. MR2748398
- [BLGGT14] T. Barnet-Lamb, T. Gee, D. Geraghty, and R. Taylor, *Potential automorphy and change of weight*, Ann. of Math. (2) **179** (2014), no. 2, 501–609. MR3152941
- [Boo19a] J. Booher, *Minimally ramified deformations when $\ell \neq p$* , Compos. Math. **155** (2019), no. 1, 1–37. MR3875451
- [Boo19b] ———, *Producing geometric deformations of orthogonal and symplectic Galois representations*, J. Number Theory **195** (2019), 115–158. MR3867437
- [BP19] J. Booher and S. Patrikis, *G -valued Galois deformation rings when $\ell \neq p$* , Math. Res. Lett. **26** (2019), no. 4, 973–990. MR4028108
- [Car12] A. Caraiani, *Local-global compatibility and the action of monodromy on nearby cycles*, Duke Math. J. **161** (2012), no. 12, 2311–2413. MR2972460
- [Car14] ———, *Monodromy and local-global compatibility for $l = p$* , Algebra Number Theory **8** (2014), no. 7, 1597–1646. MR3272276

- [Car94] H. Carayol, *Formes modulaires et représentations galoisiennes à valeurs dans un anneau local complet, p -adic monodromy and the Birch and Swinnerton-Dyer conjecture* (Boston, MA, 1991), 1994, pp. 213–237. MR1279611
- [Cas80] W. Casselman, *The unramified principal series of p -adic groups. I. The spherical function*, Compositio Math. **40** (1980), no. 3, 387–406. MR571057
- [CG13] F. Calegari and T. Gee, *Irreducibility of automorphic Galois representations of $GL(n)$, n at most 5*, Ann. Inst. Fourier (Grenoble) **63** (2013), no. 5, 1881–1912. MR3186511
- [CH13] G. Chenevier and M. Harris, *Construction of automorphic Galois representations, II*, Camb. J. Math. **1** (2013), no. 1, 53–73. MR3272052
- [CHT08] L. Clozel, M. Harris, and R. Taylor, *Automorphy for some l -adic lifts of automorphic mod l Galois representations*, Publ. Math. Inst. Hautes Études Sci. **108** (2008), 1–181. With Appendix A, summarizing unpublished work of Russ Mann, and Appendix B by Marie-France Vignéras. MR2470687
- [CLH16] A. Caraiani and B. V. Le Hung, *On the image of complex conjugation in certain Galois representations*, Compos. Math. **152** (2016), no. 7, 1476–1488. MR3530448
- [CZ21] R. Chen and J. Zou, *Local Langlands correspondence for even orthogonal groups via theta lifts*, Selecta Math. (N.S.) **27** (2021), no. 5, Paper No. 88, 71. MR4308934
- [CZ24] ———, *Arthur’s multiplicity formula for even orthogonal and unitary groups*, Preprint (2024), available at [arXiv:2103.07956](https://arxiv.org/abs/2103.07956).
- [FL82] J.-M. Fontaine and G. Laffaille, *Construction de représentations p -adiques*, Ann. Sci. École Norm. Sup. (4) **15** (1982), no. 4, 547–608. MR707328
- [Gee11] T. Gee, *Automorphic lifts of prescribed types*, Math. Ann. **350** (2011), no. 1, 107–144. MR2785764
- [Gue11] L. Guerberoff, *Modularity lifting theorems for Galois representations of unitary type*, Compos. Math. **147** (2011), no. 4, 1022–1058. MR2822860
- [Hen00] G. Henniart, *Une preuve simple des conjectures de Langlands pour $GL(n)$ sur un corps p -adique*, Invent. Math. **139** (2000), no. 2, 439–455. MR1738446
- [HSBT10] M. Harris, N. Shepherd-Barron, and R. Taylor, *A family of Calabi-Yau varieties and potential automorphy*, Ann. of Math. (2) **171** (2010), no. 2, 779–813. MR2630056
- [HT01] M. Harris and R. Taylor, *The geometry and cohomology of some simple Shimura varieties*, Annals of Mathematics Studies, vol. 151, Princeton University Press, Princeton, NJ, 2001. With an appendix by Vladimir G. Berkovich. MR1876802
- [Ish24] H. Ishimoto, *The endoscopic classification of representations of non-quasi-split odd special orthogonal groups*, Int. Math. Res. Not. IMRN **2024** (2024), no. 14, 10939–11012. MR4776199
- [Jan04] J. C. Jantzen, *Nilpotent orbits in representation theory*, Lie theory, 2004, pp. 1–211. MR2042689
- [Kon03] T. Konno, *A note on the Langlands classification and irreducibility of induced representations of p -adic groups*, Kyushu J. Math. **57** (2003), no. 2, 383–409. MR2050093
- [KSZ21] M. Kisin, S. W. Shin, and Y. Zhu, *The stable trace formula for Shimura varieties of abelian type*, Preprint (2021), available at [arXiv:2110.05381](https://arxiv.org/abs/2110.05381).
- [KT17] C. B. Khare and J. A. Thorne, *Potential automorphy and the Leopoldt conjecture*, Amer. J. Math. **139** (2017), no. 5, 1205–1273. MR3702498
- [LTX⁺24] Y. F. Liu, Y. C. Tian, L. Xiao, W. Zhang, and X. W. Zhu, *Deformation of rigid conjugate self-dual Galois representations*, Acta Math. Sin. (Engl. Ser.) **40** (2024), no. 7, 1599–1644. MR4777059
- [Pat16] S. Patrikis, *Deformations of Galois representations and exceptional monodromy*, Invent. Math. **205** (2016), no. 2, 269–336. MR3529115
- [Pen25a] H. Peng, *Fargues-Scholze parameters and torsion vanishing for special orthogonal and unitary groups*, Preprint (2025), available at [arXiv:2503.04623](https://arxiv.org/abs/2503.04623).
- [Pen25b] ———, *On the Beilinson-Bloch-Kato conjecture for polarized motives* (2025), available at [arXiv:2509.18615](https://arxiv.org/abs/2509.18615).
- [Pen26] ———, *Higher Kolyvagin theorems in the arithmetic Gross-Prasad setting*, Ph.D. thesis to appear (2026).
- [PS79] I. I. Piatetski-Shapiro, *Multiplicity one theorems, Automorphic forms, representations and L -functions* (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1, 1979, pp. 209–212. MR546599
- [Shi24] S. W. Shin, *Weak transfer from classical groups to general linear groups*, Essent. Number Theory **3** (2024), no. 1, 19–62. MR4752712
- [Sho16] J. Shotton, *Local deformation rings for GL_2 and a Breuil-Mézard conjecture when $\ell \neq p$* , Algebra Number Theory **10** (2016), no. 7, 1437–1475. MR3554238
- [Sho18] ———, *The Breuil-Mézard conjecture when $l \neq p$* , Duke Math. J. **167** (2018), no. 4, 603–678. MR3769675
- [Sol25] M. Solleveld, *On principal series representations of quasi-split reductive p -adic groups*, Pacific J. Math. **334** (2025), no. 2, 269–327. MR4862261
- [Sta18] T. Stacks Project Authors, *Stacks Project* (2018).
- [Tay08] R. Taylor, *Automorphy for some l -adic lifts of automorphic mod l Galois representations. II*, Publ. Math. Inst. Hautes Études Sci. **108** (2008), 183–239. MR2470688
- [Tho12] J. Thorne, *On the automorphy of l -adic Galois representations with small residual image*, J. Inst. Math. Jussieu **11** (2012), no. 4, 855–920. With an appendix by Robert Guralnick, Florian Herzig, Richard Taylor and Thorne. MR2979825
- [TW95] R. Taylor and A. Wiles, *Ring-theoretic properties of certain Hecke algebras*, Ann. of Math. (2) **141** (1995), no. 3, 553–572. MR1333036

- [TY07] R. Taylor and T. Yoshida, *Compatibility of local and global Langlands correspondences*, J. Amer. Math. Soc. **20** (2007), no. 2, 467–493. MR2276777
- [Wal03] J.-L. Waldspurger, *La formule de Plancherel pour les groupes p -adiques (d’après Harish-Chandra)*, J. Inst. Math. Jussieu **2** (2003), no. 2, 235–333. MR1989693
- [Whi23] D. Whitmore, *The Taylor–Wiles method for reductive groups*, Ph.D. Thesis, 2023.
- [Wil95] A. Wiles, *Modular elliptic curves and Fermat’s last theorem*, Ann. of Math. (2) **141** (1995), no. 3, 443–551. MR1333035
- [YZ25] X. Yang and X. Zhu, *On the generic part of the cohomology of shimura varieties of abelian type* (2025), available at [arXiv:2505.04329](#).
- [Zha25] X. Zhang, *Automorphic side of Taylor-Wiles method for orthogonal and symplectic groups* (2025), available at [arXiv:2411.04897](#).
- [Zhu24] X. Zhu, *A note on integral Satake isomorphisms*, Arithmetic geometry, 2024, pp. 469–489. MR4812712

HAO PENG

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA

Email: hao_peng@mit.edu

DMITRI WHITMORE

DPMMS, Wilberforce Road, Cambridge CB3 0WA, UK

Email: dw517@cam.ac.uk