FARGUES–SCHOLZE CORRESPONDENCE AND ENDOSCOPIC CLASSIFICATION FOR SPECIAL ORTHOGONAL AND UNITARY GROUPS

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Abstract. Let p be odd and let $K/\mathbb{Q}_p$ be unramified. For a special orthogonal group or a unitary group G over K that splits over an unramified extension, we prove that the Fargues–Scholze local Langlands correspondence agrees with the semisimplification of the classical correspondence for G constructed in the work of Arthur and others. As applications, we construct an unambiguous local Langlands correspondence for even special orthogonal groups, deduce the strong Kottwitz conjecture and the eigensheaf conjecture of Fargues, and establish new torsion vanishing results for orthogonal and unitary Shimura varieties.

Contents

1. Introduction 2
1.1. Torsion vanishing for special orthogonal and unitary Shimura varieties 4
1.2. An overview of the proof 7
1.3. Notation and conventions 10
2. Local Langlands correspondence via endoscopy 12
2.1. The groups 12
2.2. The L-parameters 16
2.3. The correspondence 19
2.4. Compatibility with parabolic induction 23
2.5. Supercuspidal L-packets 26
2.6. Combinatorics on L-parameters 26
3. Local Langlands correspondence via moduli of local shtukas 28
3.1. The correspondence 28
3.2. Local shtuka spaces 32
3.3. Spectral actions 34
3.4. Weak version of the Kottwitz conjecture 35
4. Cohomology of orthogonal and unitary Shimura varieties 40
4.1. The groups 40
4.2. Endoscopic classification of automorphic representations 42
4.3. Controlled cuspidal automorphic representations 44
4.4. The Shimura data 46
4.5. Langlands–Kottwitz method 50
5. Local and global Shimura varieties 56
5.1. Basic uniformization of Shimura varieties 56
5.2. Globalization 58
6. Proof of the compatibility property 60
6.1. The first case 62
6.2. The second case 62
7. Applications 63

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7.1. Unambiguous local Langlands correspondence for even orthogonal groups 63
7.2. Naturality of Fargues–Scholze LLC 68
7.3. Strong Kottwitz conjecture 70
8. A vanishing result for torsion cohomology of Shimura varieties 76
8.1. Generic semisimple L-parameters 77
8.2. Perverse t-exactness and vanishing results 81
Appendix A. (Twisted) Endoscopy theory 86
A.1. Endoscopic triples 87
A.2. Transfer of orbital integrals 87
A.3. Cuspidal functions 88
A.4. Simple stable trace formulas 89
Acknowledgements 92
References 92

1. Introduction

For a connected reductive group G over a finite extension K of $\mathbb{Q}_p$ for some rational prime p, the conjectural local Langlands correspondence is a map from the set $\Pi(G)$ of isomorphism classes of irreducible admissible representations of $G(K)$ to the set $\Phi(G)$ of conjugacy classes of L-parameters[1]

$$\phi : W_K \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G,$$

which should have finite fibers called L-packets and satisfy various properties; see [Bor79]. When G is a special orthogonal group or a unitary group over K, such a map is constructed by Arthur [Art13], Chen–Zou [CZ21a], and Ishimoto [Ish24] in the special orthogonal case, and by Mok [Mok15] and Kaletha–Minguez–Shin–White [KMSW14] in the unitary case. When G is a special orthogonal group associated with a quadratic space of even dimension $2n$, the correspondence is only well-defined up to $\mathrm{O}_{2n}(\mathbb{C})$-conjugation. In this case, set

$$\tilde{\Phi}(G) := \Phi(G)/\mathrm{O}_{2n}(\mathbb{C}).$$

Also, when $G = \mathrm{GSpin}(V)$ for a quadratic space V over K, the correspondence is constructed in [GT19, Theorem 2.6.1] for those representations π whose central character is the square of another character. These constructions ultimately rely on endoscopy and trace formula techniques. We denote the resulting correspondence by

$$\mathrm{rec}_G : \Pi(G) \longrightarrow \begin{cases} \Phi(G), & \text{outside the even special orthogonal case,} \\ \tilde{\Phi}(G), & \text{in the even special orthogonal case.} \end{cases}$$

On the other hand, fix a rational prime $\ell \neq p$ and an isomorphism $\iota_\ell : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}_\ell}$. Using excursion operators on the moduli stack of G-bundles on the Fargues–Fontaine curve, Fargues and Scholze [FS24] constructed a candidate for a semisimplified local Langlands correspondence for every connected reductive group G, namely,

$$\mathrm{rec}^{\mathrm{FS}}_G : \Pi(G) \to \Phi^{\mathrm{ss}}(G), \qquad \pi \mapsto \iota_\ell^{-1}\phi^{\mathrm{FS}}_{\iota_\ell \pi}.$$

Here $\Phi^{\mathrm{ss}}(G)$ denotes the set of conjugacy classes of continuous semisimple homomorphisms

$$\phi : W_K \to {}^L G$$

whose composition with the canonical projection ${}^L G \to W_K$ is id_{W_K}. Moreover, $\mathrm{rec}^{\mathrm{FS}}_G$ satisfies some desired properties listed in [FS24, Theorem 1.9.6]. It is known that $\mathrm{rec}^{\mathrm{FS}}_G$ is independent of the choice of ℓ; see [Sch26].

[1]Here and throughout the paper we take the Langlands L-group ${}^L G$ in the Weil form.

It is both natural and nontrivial to ask whether rec_G and $\mathrm{rec}_G^{\mathrm{FS}}$ are compatible when they both exist, in the sense that there exists a commutative diagram

$$\begin{tikzcd} \Pi(G) \arrow[r, "\mathrm{rec}_G"] \arrow[dr, "\mathrm{rec}_G^{\mathrm{FS}}"'] & \Phi(G) \arrow[d, "(-)^{\mathrm{ss}}"] \\ & \Phi^{\mathrm{ss}}(G) \end{tikzcd}, \tag{1.1}$$

where $(-)^{\mathrm{ss}}$ precomposes a parameter $\phi \in \Phi(G)$ with the map

$$W_K \to W_K \times \mathrm{SL}_2(\mathbb{C}) : g \mapsto \left(g, \begin{bmatrix} |g|_K^{1/2} & 0 \\ 0 & |g|_K^{-1/2} \end{bmatrix} \right).$$

Here $|-|_K$ is defined to be the composition $W_K \to W_K^{\mathrm{ab}} \xrightarrow{\mathrm{Art}_K^{-1}} K^\times \xrightarrow{|-|_K} \mathbb{R}_+$.

Our main result is the following theorem on compatibility of the Fargues–Scholze local Langlands correspondence with the "classical local Langlands correspondence" defined in [Art13], [Mok15], [KMSW14], [CZ21a], [CZ21b], and [Ish24]:

Theorem A. *Suppose $p > 2$ and $K/\mathbb{Q}_p$ is unramified.*

(1) If $G = \mathrm{U}(V)$ where V is a Hermitian space with respect to the unramified quadratic extension K_1/K, then the diagram (1.1) is commutative.

(2) If $G = \mathrm{SO}(V)$ where V is a quadratic space over K with $\dim(V) = 2n+1$ for some positive integer n, then the diagram (1.1) is commutative.

(3) If $G = \mathrm{SO}(V)$ where V is a quadratic space over K of dimension $2n$ for some positive integer n such that G splits over an unramified quadratic extension of K (equivalently, $\mathrm{ord}_K(\mathrm{disc}(V)) \equiv 0 (\mathrm{mod}\ 2)$; see §2.1), then the diagram (1.1) is commutative up to conjugation by $\mathrm{O}(2n, \mathbb{C})$.

Remark. We have the following remarks concerning Theorem A.

(1) When G is an inner form of a general linear group over any finite extension of $\mathbb{Q}_p$, Hansen, Kaletha and Weinstein established in [HKW22, Theorem 6.6.1] that the Fargues–Scholze LLC coincides with the usual (semisimplified) LLC for inner forms of general linear groups given in [DKV84, Rog83].

(2) When G is an inner form of Sp_4 or GSp_4 over an unramified finite extension of $\mathbb{Q}_p$ for $p > 2$, the compatibility is established in [Ham25].

(3) For $G = \mathrm{U}(V)$ or $\mathrm{GU}(V)$ where V is an odd-dimensional Hermitian space with respect to the unramified quadratic extension $\mathbb{Q}_{p^2}/\mathbb{Q}_p$, the compatibility is established in [MHN24, Theorem 1.1], and their proof is different from ours. In fact, they established the Kottwitz conjecture first by proving an averaging formula of Shin for $\mathrm{GU}(V)$, and they restricted to the case $K = \mathbb{Q}_p$ because the Hasse principle holds for unitary similitude groups over $\mathbb{Q}$.

(4) The assumptions that $p > 2$ and that the relevant splitting extension over $\mathbb{Q}_p$ is unramified are necessary in order to apply Shen's result [She20] that the relevant local shtuka spaces uniformize Shimura varieties of Abelian type. These assumptions in Theorem A and its corollaries can be removed for such G once the main result of [She20] is established for a Shimura datum $(\mathbb{G}, \{\mu\})$ such that $\mathbb{G} \otimes \mathbb{Q}_p \cong \mathrm{Res}_{K/\mathbb{Q}_p} G$. During the review of this paper, we learned that [DvHKZ26] had removed these assumptions, building on computations carried out in the present work.

Theorem A is proved in §6. Moreover, in the third case of Theorem A (i.e., even special orthogonal groups), we use the compatibility property to construct an unambiguous version of the local Langlands correspondence for G, eliminating the ambiguity up to outer automorphisms by requiring compatibility with the Fargues–Scholze local Langlands correspondence, which is defined canonically without outer automorphisms.

Theorem B. *In the third case of* Theorem A, *there exists a map*

$$\mathrm{rec}_G^\natural : \Pi(G) \longrightarrow \Phi(G)$$

lifting the correspondence defined in Arthur [Art13] *and Chen–Zou* [CZ21a]*. It is compatible with the Fargues–Scholze local Langlands correspondence in the sense that diagram* (1.1) *is commutative. Moreover,* $\mathrm{rec}_G^\natural$ *satisfies the relevance, temperedness, discreteness, internal parametrization, Langlands quotient, standard* γ*-factor, Plancherel measure, and local intertwining properties stated precisely in* Theorem 7.1.1*. In particular, Vogan's version of the local Langlands conjecture* [Vog93] *holds for unramified even special orthogonal groups.*

Theorem B is proved in Theorem 7.1.1.

Using the unambiguous local Langlands correspondence, we verify in §7.2 the naturality property of the Fargues–Scholze local Langlands correspondence for those G appearing in Theorem A, thereby confirming [Ham24, Assumption 6.5]. We also establish a weaker result for a central extension of $\mathrm{Res}_{K/\mathbb{Q}_p} G$, which will be used to deduce a torsion vanishing result for suitable Shimura varieties.

We next show that the classical Langlands correspondence, together with geometric techniques, provides sufficient input to verify part of the categorical local Langlands conjecture of Fargues–Scholze [FS24, Conjecture X.1.4].

Theorem C. *Suppose* G *is one of the groups appearing in* Theorem A*,* $p > 2$*,* $K/\mathbb{Q}_p$ *is unramified, and* $\phi \in \Phi(G)$ *is supercuspidal.*

(1) The sheaf

$$\mathcal{G}_\phi = \bigoplus_{b \in B(G)_{\mathrm{bas}}} \bigoplus_{\pi_b \in \Pi_\phi(G_b)} i_{b!}(\pi_b) \in \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}_\ell})$$

admits an action of $\mathfrak{S}_\phi := Z_{\widehat{G}}(\phi)$ *satisfying conditions (i)–(iv) of Fargues' conjecture* [Far16, Conjecture 4.4] *for* G*. In particular,* $\mathcal{G}_\phi$ *is a Hecke eigensheaf for* ϕ*.*

(2) The strong Kottwitz conjecture [HKW22, Conjecture 1.0.1] *holds for* G *and any conjugacy class of geometric cocharacters* $\{\mu\}$ *for* $G_{\overline{K}}$*, up to a reparametrization of elements of the* L*-packet of* ϕ *that is independent of the choice of* $\{\mu\}$*. Moreover, assuming* Hypothesis 4.5.8*, this reparametrization is trivial.*

Remark. The strong Kottwitz conjecture was previously shown when G is an unramified odd unitary group over $\mathbb{Q}_p$ [MHN24], and a weaker version is known when G is an inner form of GSp_4 or $G = \mathrm{Sp}_4$ over an unramified extension of $\mathbb{Q}_p$ [Ham25]. Our proof uses a new globalization method that connects the cohomology of local shtuka spaces with the endoscopic part of the cohomology of Shimura varieties.

Hypothesis 4.5.8 describes the Galois representation appearing in the generic part of the middle degree ℓ-adic cohomology of orthogonal and unitary Shimura varieties. It is known for orthogonal Shimura varieties over $\mathbb{Q}$ by [Zhu24, Corollary 9.8.10]. In particular, Theorem C proves the strong form of the Kottwitz conjecture for special orthogonal groups over $\mathbb{Q}_p$ for $p > 2$, generalizing [Ham25, Theorem 1.3]. In general, Hypothesis 4.5.8 will follow from a sequel to [KSZ21] as long as the full endoscopic classification for orthogonal and unitary groups is obtained.

Theorem C is proved in Theorems 7.3.4, 7.3.5, and 7.3.7.

1.1. **Torsion vanishing for special orthogonal and unitary Shimura varieties.** We use the compatibility result to establish new torsion vanishing results for Shimura varieties of orthogonal or unitary type. We now introduce the necessary background and notation. Let $\mathbb{G}$ be a connected reductive group over $\mathbb{Q}$ with a Shimura datum $(\mathbb{G}, \mathbb{X})$, and let $E \subset \mathbb{C}$ be the associated reflex field. Fix an odd rational prime p that is coprime to $\#\pi_1([\mathbb{G}, \mathbb{G}])$, together with an isomorphism

$\iota_p : \mathbb{C} \to \overline{\mathbb{Q}_p}$ inducing an embedding $E \to \overline{\mathbb{Q}_p}$. We denote by G the base change of $\mathbb{G}$ to $\mathbb{Q}_p$. We assume that G is unramified and equipped with a Borel pair (B, T) and a hyperspecial subgroup $\mathscr{K}_p$ of $\mathsf{G}(\mathbb{Q}_p)$. Let $\mathscr{K}^p \leq \mathbb{G}(\mathbf{A}_f^p)$ be a compact open subgroup such that $\mathscr{K} := \mathscr{K}_p\mathscr{K}^p \leq \mathbb{G}(\mathbf{A}_f)$ is neat. Let ℓ be a rational prime that is coprime to $p \cdot \#\pi_0(Z(\mathsf{G}))$, and let the coefficient field Λ be either $\overline{\mathbb{Q}_\ell}$ or $\overline{\mathbb{F}_\ell}$.

Definition ([HL24, Definition 6.2])**.** Suppose G is an arbitrary quasisplit reductive group over a finite extension $K/\mathbb{Q}_p$ with a Borel pair (B, T), and $\phi_{\mathsf{T}} \in \Phi^{\mathrm{ss}}(\mathsf{T}, \Lambda)$ is a semisimple L-parameter. Let $\phi_{\mathsf{T}}^\vee$ denote the Chevalley dual of ϕ_{T}. Then ϕ_{T} is said to be *generic* (or of Langlands–Shahidi type) if for every positive coroot $\mu \in \Phi^\vee(\mathsf{G}, \mathsf{T})^+ \subset X_\bullet(\mathsf{T})$, the following Galois cohomology complexes vanish:

$$\mathrm{R\Gamma}(W_K, {}^L\mathcal{T}_\mu \circ \phi_{\mathsf{T}}), \quad \mathrm{R\Gamma}(W_K, {}^L\mathcal{T}_\mu \circ \phi_{\mathsf{T}}^\vee).$$

Here ${}^L\mathcal{T}_\mu$ denotes the extended highest weight tilting module of ${}^L\mathsf{T}$ with Λ-coefficients associated with μ; see (3.6).

We now state the following torsion-vanishing conjecture for Shimura varieties.

Conjecture ([Car23, HL24])**.** Suppose $\phi \in \Phi^{\mathrm{ss}}(\mathsf{G}, \overline{\mathbb{F}_\ell})$ is an unramified, semisimple, toral generic L-parameter, corresponding via the Satake isomorphism to a maximal ideal $\mathfrak{m} \subset \overline{\mathbb{F}_\ell}[\mathscr{K}_p \backslash \mathsf{G}(\mathbb{Q}_p)/\mathscr{K}_p]$. Then the complex $\mathrm{R\Gamma}_c(\mathrm{Sh}_{\mathscr{K}}(\mathbb{G}, \mathbb{X})_{\overline{E}}, \overline{\mathbb{F}_\ell})_{\mathfrak{m}}$ (resp. $\mathrm{R\Gamma}(\mathrm{Sh}_{\mathscr{K}}(\mathbb{G}, \mathbb{X})_{\overline{E}}, \overline{\mathbb{F}_\ell})_{\mathfrak{m}}$) is concentrated in degrees $0 \leq i \leq \dim_{\mathbb{C}}(\mathbb{X})$ (resp. $\dim_{\mathbb{C}}(\mathbb{X}) \leq i \leq 2\dim_{\mathbb{C}}(\mathbb{X})$).

This torsion vanishing conjecture has been established in [CS17, CS24, Kos21] and [HL24] in the case where $(\mathbb{G}, \mathbb{X})$ is a PEL-type Shimura datum of type A or C_2 and G is a product of certain groups related either to GL_n over an unramified extension of $\mathbb{Q}_p$, or U_{2k+1} with respect to $\mathbb{Q}_{p^2}/\mathbb{Q}_p$, or U_2 with respect to a quadratic extension of unramified extensions of $\mathbb{Q}_p$, with p and ℓ satisfying certain properties.

In this work, we extend the list of known cases, particularly when $(\mathbb{G}, \mathbb{X})$ is an orthogonal or unitary Shimura datum associated with a quadratic or Hermitian space over a totally real number field F with standard indefinite signature, G is among the unramified groups listed in Theorem A, and ℓ is sufficiently large.

Remark. The above torsion vanishing conjecture was established in [DvHKZ24] in the case where $\mathsf{G} = \mathbb{G} \otimes \mathbb{Q}_p$ is split and the Shimura variety is compact of Hodge type, under the hypothesis that the Fargues–Scholze correspondence for G is compatible with the so-called classical local Langlands correspondence; see [Ham24, Assumption 6.5]. However, their result does not apply directly to the orthogonal Shimura variety $\mathrm{Sh}(\mathrm{Res}_{F/\mathbb{Q}} \mathrm{SO}(\mathbf{V}), \mathbf{X})$ when the quadratic space $\mathbf{V}$ over a totally real field F has large rank, because it is not of Hodge type. A natural approach is to consider a Hodge type Shimura datum $(\mathbf{G}^\sharp, \mathbf{X}^\sharp)$ with a map of Shimura data $(\mathbf{G}^\sharp, \mathbf{X}^\sharp) \to (\mathrm{Res}_{F/\mathbb{Q}} \mathrm{SO}(\mathbf{V}), \mathbf{X})$ such that the morphism $\mathbf{G}^\sharp \to \mathrm{Res}_{F/\mathbb{Q}} \mathrm{SO}(\mathbf{V})$ is a central extension. However, $\mathbf{G}^\sharp$ has derived subgroup $\mathrm{Res}_{F/\mathbb{Q}} \mathrm{Spin}(\mathbf{V})$, for which the so-called classical local Langlands correspondence has not yet been constructed. In this work, we modify the argument of [DvHKZ24] and weaken the hypothesis of [Ham24, Assumption 6.5] to our Hypothesis D below. We then establish the torsion vanishing for $\mathrm{Sh}(\mathbf{G}^\sharp, \mathbf{X}^\sharp)$ and use the Hochschild–Serre spectral sequence to deduce the torsion vanishing for $\mathrm{Sh}(\mathrm{Res}_{F/\mathbb{Q}} \mathrm{SO}(\mathbf{V}), \mathbf{X})$. We also treat certain cases where p is not split in F.

If G_{ad} is a product of Weil restrictions of split simple groups, then there exists a simpler proof without using geometric Eisenstein series. In this introduction, we state a more general result, which applies to the above-mentioned orthogonal or unitary case by constructing a central extension in §4.1. We first state a hypothesis on the Fargues–Scholze local Langlands correspondence:

Hypothesis D. Suppose G is a quasisplit connected reductive group over a p-adic number field K with a Borel pair (B, T) and $\phi \in \Phi^{\mathrm{ss}}(\mathsf{G}, \overline{\mathbb{Q}_\ell})$ is a semisimple generic toral L-parameter. For any

$\mathsf{b} \in B(\mathsf{G})$ and any $\rho \in \Pi(\mathsf{G}_{\mathsf{b}}, \overline{\mathbb{Q}_\ell})$, if the composition

$$W_K \xrightarrow{\phi_\rho^{\mathrm{FS}}} {}^L\mathsf{G}_{\mathsf{b}}(\overline{\mathbb{Q}_\ell}) \longrightarrow {}^L\mathsf{G}(\overline{\mathbb{Q}_\ell}),$$

where the second arrow is the twisted embedding defined in [FS24, §IX.7.1], equals ϕ, then b is unramified.

By our main theorem and [Ham24, Lemma 3.14], this hypothesis holds for those groups G/K appearing in Theorem A that are quasisplit.

We now state our main theorem on torsion vanishing for certain Shimura varieties of Abelian type. This theorem is proved in Theorem 8.2.10.

Theorem E. *Suppose that the following assumptions hold:*

(1) $\mathrm{Sh}_{\mathcal{K}}(\mathbb{G}, \mathbb{X})$ *is proper, and there exists a Shimura datum of Hodge type* $(\mathbb{G}^\sharp, \mathbb{X}^\sharp)$*, and a morphism of Shimura data* $(\mathbb{G}^\sharp, \mathbb{X}^\sharp) \to (\mathbb{G}, \mathbb{X})$ *such that* $\mathbb{G}^\sharp_{\mathrm{ad}} \to \mathbb{G}_{\mathrm{ad}}$ *is an isomorphism.*

(2) There is an isomorphism

$$\mathsf{G}_{\mathrm{ad}} \cong \prod_{i=1}^k \mathrm{Res}_{L_i/\mathbb{Q}_p} \mathsf{H}_i,$$

where each H_i *is a split simple group over* L_i*. Via* ι_p*, the conjugacy class of Hodge cocharacters associated with* $\mathbb{X}^\sharp$ *induces a dominant cocharacter* $\mu_{\mathrm{ad}} = (\mu_1, \ldots, \mu_k)$ *of* $\mathsf{G}_{\mathrm{ad}, \overline{\mathbb{Q}_p}}$*. Under the canonical decomposition*

$$\left(\mathrm{Res}_{L_i/\mathbb{Q}_p} \mathsf{H}_i\right)_{\overline{\mathbb{Q}_p}} \cong \prod_{\tau \in \mathrm{Hom}_{\mathbb{Q}_p}(L_i, \overline{\mathbb{Q}_p})} \left(\mathsf{H}_i \otimes_{L_i, \tau} \overline{\mathbb{Q}_p}\right),$$

each μ_i *is trivial on all but at most one simple factor.*

(3) ℓ *is a rational prime that is coprime to* $p \cdot \#\pi_0(Z(\mathsf{G}^\sharp)) \cdot \#\pi_0(Z(\mathsf{G}))$*, and* $\mathfrak{m}$ *is a maximal ideal of the* ℓ*-torsion Hecke algebra* $\mathcal{H}_{\mathcal{K}_p} := \overline{\mathbb{F}_\ell}[\mathcal{K}_p \backslash \mathsf{G}(\mathbb{Q}_p)/\mathcal{K}_p]$*.*

If the semisimple toral L-parameter $\phi_{\mathfrak{m}}$ *corresponding to* $\mathfrak{m}$ *is generic and* Hypothesis D *holds for every semisimple generic toral lift of* $\phi_{\mathfrak{m}}$*, then* $\mathrm{H}^i_{\mathrm{\acute{e}t}}(\mathrm{Sh}_{\mathcal{K}}(\mathbb{G}, \mathbb{X})_{\overline{E}}, \overline{\mathbb{F}_\ell})_{\mathfrak{m}}$ *vanishes unless* $i = \dim_{\mathbb{C}}(\mathbb{X})$*.*

Remark. The hypothesis that $\mathrm{Sh}_{\mathcal{K}}(\mathbb{G}, \mathbb{X})$ is proper is expected to be unnecessary, once one has constructed the minimally compactified Igusa stack for $\mathrm{Sh}(\mathbb{G}^\sharp, \mathbb{X}^\sharp)$ in the sense of [Zha23] and compared the fibers of the Hodge–Tate map on it with the minimally compactified Igusa varieties. This has been done when $\mathrm{Sh}(\mathbb{G}^\sharp, \mathbb{X}^\sharp)$ is of PEL type A or C in [HL24].

In particular, this theorem generalizes previous results of [CS17, Kos21, CS24, HL24] to compact orthogonal and unitary Shimura varieties. This is because we may construct a central extension $\mathbf{G}^\sharp$ of $\mathbf{G} := \mathrm{Res}_{F/\mathbb{Q}} \mathrm{U}(\mathbf{V})^\circ$ with a morphism of Shimura data $(\mathbf{G}^\sharp, \mathbf{X}^\sharp) \to (\mathbf{G}, \mathbf{X})$, so that $(\mathbf{G}^\sharp, \mathbf{X}^\sharp)$ defines a Shimura datum of Hodge type. In the unitary case, we take

$$\mathbf{G}^\sharp = \mathbf{G} \times \mathbf{Z}^{\mathbb{Q}}.$$

Here

$$\mathbf{Z}^{\mathbb{Q}} = \{z \in \mathrm{Res}_{F_1/\mathbb{Q}} \mathrm{GL}_1 : \mathrm{Nm}_{F_1/F}(z) \in \mathbb{Q}^\times\}$$

and F_1/F is the CM-extension associated with the Hermitian space $\mathbf{V}$. The desired map of Shimura data $(\mathbf{G}^\sharp, \mathbf{X}^\sharp) \to (\mathbf{G}, \mathbf{X})$ is constructed by Rapoport, Smithling, and Zhang in [RSZ20]. In the orthogonal case, following Carayol [Car86, p. 163], we construct, for each $\daleth \in \mathbb{R}_+ \mathrm{i}$ such that $\mathbb{Q}(\daleth)$ is an imaginary quadratic field, a group $\mathbf{G}^\sharp$ fitting into an exact sequence

$$1 \to \mathbf{Z}^{\mathbb{Q}} \to \mathbf{G}^\sharp \to \mathbf{G} \to 1,$$

where

$$\mathbf{Z}^{\mathbb{Q}} = \{z \in \mathrm{Res}_{F(\daleth)/\mathbb{Q}} \mathrm{GL}_1 : \mathrm{Nm}_{F(\daleth)/F}(z) \in \mathbb{Q}^\times\}.$$

If moreover we assume that p is unramified in F and $\mathbb{Q}(\daleth)/\mathbb{Q}$ is split at p, then there exists an isomorphism

$$\mathbf{G}^\sharp \otimes \mathbb{Q}_p \cong \mathrm{GL}_1 \times \mathrm{Res}_{F\otimes\mathbb{Q}_p/\mathbb{Q}_p} \mathrm{GSpin}(\mathbf{V} \otimes \mathbb{Q}_p).$$

In fact, proving this torsion vanishing result for orthogonal Shimura varieties is one of the main motivations for this paper. In Euler system arguments via "level-raising congruences" in higher-dimensional Shimura varieties, as pioneered by Bertolini and Darmon [BD05] for Shimura curves, we need to construct elements in the cohomology of Shimura varieties via the Jacquet–Langlands correspondence. The natural way to do so is to take the Abel–Jacobi map of a globally defined cycle that is cohomologically trivial. The point is that a certain Hecke translate of a global cycle class becomes cohomologically trivial if the target cohomology group is itself trivial. For example, this strategy was applied in [Liu16, Liu19, LT20, LTX+22] and will be used in the author's upcoming work [Pen26] on higher-dimensional analogues of Kolyvagin-type theorems for product of Shimura varieties of orthogonal type in the arithmetic Gross–Prasad setting. For further applications of torsion vanishing results, we refer the reader to Caraiani's ICM report [Car23].

1.2. **An overview of the proof.** We summarize the proof of Theorem A, adapting the method of Hamann [Ham25]. We exclude the case of even orthogonal groups in this introduction, as the additional outer automorphism complicates the notation. The proof proceeds by induction on the geometric rank of G, with low-rank cases verified by direct inspection. For higher ranks, if $\pi \in \Pi(G)$ is non-supercuspidal, we invoke the induction hypothesis along with the compatibility of rec_G (resp. $\mathrm{rec}_G^{\mathrm{FS}}$) with parabolic induction, since any proper Levi subgroup of G is a product of (Weil restrictions of) general linear groups and a group of the same type as G, but with smaller geometric rank. We may therefore assume that π is supercuspidal. We establish the compatibility for pure inner forms of G simultaneously. If ϕ is the classical L-parameter of π and $\Pi_\phi(G^*)$ contains a non-supercuspidal representation ρ_{nsc}, then the compatibility is already known for ρ_{nsc}. We then propagate this property to other representations in the L-packets of pure inner forms of G with classical parameter ϕ. The crucial input is a description of the cohomology of the local shtuka spaces $\mathrm{Sht}_{G,b,\{\mu\}}$ defined in [SW20], where $\{\mu\}$ is a geometric conjugacy class of G related to the Hodge cocharacter of a suitable global Shimura variety of orthogonal or unitary type, and $b \in B(G, \{\mu\})$ is the unique nontrivial basic element. This local shtuka space carries an action of $G_b(K) \times G(K) \times W_{E_{\{\mu\}}}$, where $E_{\{\mu\}}/K$ is the reflex field of $\{\mu\}$. For any $\rho \in \Pi_\phi(G_b)$, the complex $\mathrm{R\Gamma}_c^\flat(G, b, \{\mu\})[\rho]$ is isomorphic to the image of ρ under the relevant Hecke operator. Since Hecke operators and excursion operators commute, it follows that any representation of $G(K)$ occurring in $\mathrm{R\Gamma}_c^\flat(G, b, \{\mu\})[\rho]$ has Fargues–Scholze parameter equal to that of ρ.

To analyze which representations of G appear, we use the weak Kottwitz conjecture established by Hansen, Kaletha and Weinstein [HKW22].[2] In fact, π does not necessarily appear in the complex $\mathrm{R\Gamma}_c^\flat(G, b, \{\mu\})[\rho_{\mathrm{nsc}}]$, but we may iterate this process, replacing ρ_{nsc} by those representations that appear, until π eventually appears. This ultimately depends on a detailed analysis of the combinatorics of the centralizer of the L-parameter ϕ in $\widehat{G}$, see §2.6.

We are now left with the case in which $\Pi_\phi(G^*)$ consists entirely of supercuspidal representations. A result of Mœglin and Tadić, recalled in Proposition 2.5.1, then implies that ϕ is supercuspidal; in particular, it is discrete and trivial on the $\mathrm{SL}_2(\mathbb{C})$-factor. Since Hecke operators commute with the excursion algebra, we may reduce to the quasisplit case, and it suffices to prove compatibility for each $\rho \in \Pi_\phi(G_b)$; see [Ham25, Lemma 3.15].

[2] We remark that when G is an even special orthogonal group, the hypothesis in [HKW22] remains unproven, since only a version of the local Langlands correspondence up to conjugacy by the full orthogonal group is available. Instead, we use the weak endoscopic character identity established in [Pen25] and modify the arguments therein to establish a weaker version of the weak Kottwitz conjecture, Theorem 3.4.1, valid up to conjugacy by the full orthogonal group. This weaker version is enough for the argument to work.

Let $\widehat{\mathrm{Std}}$ denote the standard representation of $\widehat{G} \rtimes \mathrm{Gal}(K'/K_1)$, where K' is the splitting field of G. Because ϕ is discrete, we may write

$$\widehat{\mathrm{Std}} \circ \phi|_{W_{K_1}} = \phi_1 + \cdots + \phi_r,$$

where the ϕ_i are pairwise nonisomorphic irreducible representations, each occurring with multiplicity one. Hamann [Ham25] and Koshikawa [Kos21] showed that if every ϕ_i occurs as a subquotient of

$$\bigoplus_{\rho' \in \Pi_\phi(G_b)} \mathrm{R\Gamma}_c^\flat(G, b, \{\mu\})[\rho']$$

as a W_{K_1}-representation, then every ϕ_i occurs in $\widehat{\mathrm{Std}} \circ \phi_\rho^{\mathrm{FS}}|_{W_{K_1}}$. The latter representation is semisimple and has dimension $\sum_{i=1}^r \dim(\phi_i)$. Thus the containment of all the pairwise nonisomorphic constituents forces

$$\widehat{\mathrm{Std}} \circ \phi_\rho^{\mathrm{FS}}|_{W_{K_1}} \cong \widehat{\mathrm{Std}} \circ \phi|_{W_{K_1}}.$$

The results of [GGP12] then imply $\phi_\rho^{\mathrm{FS}} = \phi$.

Thus it suffices to prove that

$$\bigoplus_{\rho' \in \Pi_\phi(G_b)} \mathrm{R\Gamma}_c^\flat(G, b, \{\mu\})[\rho']$$

admits a subquotient isomorphic to $\widehat{\mathrm{Std}} \circ \phi$ as a W_{K_1}-representation. This is where global inputs become necessary, i.e., by relating this complex to the cohomology of a relevant global Shimura variety. The cohomology of the relevant global Shimura variety is studied via the Langlands–Kottwitz method and related to automorphic forms, whose local components are governed by classical L-parameters via Arthur's multiplicity formula. To be more precise, we elaborate on the case when G is an odd special orthogonal group as an example. According to a result of Shen [She20], the local shtuka space uniformizes the basic Newton stratum of the generic fiber of a relevant Shimura variety, as defined in [CS17]. A relevant Shimura variety is given by $(\mathbf{G}, \mathbf{X})$, where $\mathbf{G}$ is a standard indefinite special orthogonal group over a totally real field F with p inert and $F_p \cong K$, such that $\mathbf{G} \otimes_F K \cong G$. Let $\mathbf{A}_F$ and $\mathbf{A}_{F,f}$ denote the ring of adeles and finite adeles of F, respectively, and let $\mathcal{K}_p \leq \mathbf{G}(K)$ and $\mathcal{K}^p \leq \mathbf{G}(\mathbf{A}_{F,f}^p)$ be sufficiently small level subgroups, thereby yielding the adic Shimura variety $\mathcal{S}_{\mathcal{K}_p\mathcal{K}^p}(\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G}, \mathbf{X})$ defined over $\mathbb{C}_p$. For an algebraic representation ξ of $\left(\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G}\right)_{\mathbb{C}}$ with sufficiently regular highest weight, let $\mathcal{L}_{\iota_\ell\xi}$ be the associated $\overline{\mathbb{Q}_\ell}$-local system on $\mathcal{S}_{\mathcal{K}_p\mathcal{K}^p}(\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G}, \mathbf{X})$, and consider the cohomology

$$\mathrm{R\Gamma}_c(\mathcal{S}_{\mathcal{K}_p\mathcal{K}^p}(\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G}, \mathbf{X}), \mathcal{L}_{\iota_\ell\xi}).$$

The basic uniformization result of Shen implies a $G(K) \times W_K$-equivariant map

$$\begin{aligned}\Theta : \mathrm{R\Gamma}_c(G, b, \mathbf{1}, \{\mu\}) \otimes \iota_\ell \left|-\right|_{K_1}^{\frac{-\dim_{\mathbb{C}}(\mathbf{X})}{2}} [\dim_{\mathbb{C}}(\mathbf{X})] \otimes^{\mathrm{L}}_{G_b^*(K)} \mathcal{A}\left(\mathbf{G}'(F)\backslash\mathbf{G}'(\mathbf{A}_{F,f})/\mathcal{K}^p, \mathscr{L}_{\iota_\ell\xi}\right) \\ \longrightarrow \mathrm{R\Gamma}_c\left(\mathcal{S}(\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G}, \mathbf{X})_{\mathcal{K}^p}, \mathcal{L}_{\iota_\ell\xi}\right),\end{aligned}$$

where $\mathbf{G}'$ is an inner form of $\mathbf{G}$ satisfying $\mathbf{G}' \otimes_F K \cong G_b^*$ and such that $\mathbf{G}'(F \otimes \mathbb{R})$ is compact. Here

$$\mathcal{A}(\mathbf{G}'(F)\backslash\mathbf{G}'(\mathbf{A}_{F,f})/\mathcal{K}^p, \mathscr{L}_{\iota_\ell\xi})$$

denotes the space of $\mathcal{K}^p$-invariant algebraic automorphic forms valued in $\iota_\ell\xi$. Next, we note that the pair $(\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G}, \{\mu_{\mathrm{Hdg}}\})$ is fully Hodge-Newton reducible as defined in [GHN19], where μ_{Hdg} is the Hodge cocharacter associated with $\mathbf{X}$. This implies that the flag variety $\mathrm{Gr}_{\mathrm{Res}_{K/\mathbb{Q}_p}G, \{\mu_{\mathrm{Hdg}}\}}$ is parabolically induced as a $G(K)$-space. This is called "Boyer's trick". Using the Hodge–Tate period map

$$\pi_{\mathrm{HT}} : \mathcal{S}_{\mathcal{K}^p}(\mathbf{G}, \mathbf{X}) \to \mathrm{Gr}_{\mathrm{Res}_{K/\mathbb{Q}_p}G, \{\mu_{\mathrm{Hdg}}\}},$$

defined in [CS17], when we restrict to the summands on both sides of Θ where $G(K)$ acts by a supercuspidal representation, we obtain a $G(K)\times W_K$-equivariant isomorphism, which is also functorial with respect to $\mathcal{K}^p$.

We globalize the given $\rho\in\Pi(G_b)$ to a cuspidal automorphic representation Π' of $\mathbf{G}'$, such that the $\mathcal{K}^p$-fixed subspace of Π' occurs as a $G_b(K)$-stable direct summand of

$$\mathcal{A}\left(\mathbf{G}'(F)\backslash\mathbf{G}'(\mathbf{A}_{F,f})/\mathcal{K}^p,\mathscr{L}_{\iota_\ell\xi}\right)$$

for some ξ with sufficiently regular highest weight, where

- Π' is an unramified twist of the Steinberg representation at some nonempty subset Σ^{St} of places of F,
- Π' is supercuspidal at some nonempty subset Σ^{sc} of finite places of F disjoint with Σ^{St}, and there exists $v\in\Sigma^{\mathrm{sc}}$ such that Π'_v has a simple supercuspidal L-parameter ϕ_v, meaning that $\widehat{\mathrm{Std}}\circ\phi_v$ is irreducible as a representation of W_{F_v}.
- Π' is unramified outside some nonempty subset Σ of places of F containing $\Sigma^{\mathrm{sc}}\cup\Sigma^{\mathrm{St}}\cup\Sigma_F^\infty$, and $\mathcal{K}^p$ decomposes as $\mathcal{K}^p=\mathcal{K}_{\Sigma\smallsetminus\{p\}}\mathcal{K}^\Sigma$.

These conditions ensure that Arthur's multiplicity formula can be applied to analyze the cuspidal automorphic representations $\dot{\Pi}'$ in the near equivalence class of Π'. In particular, for each such $\dot{\Pi}'$, $\dot{\Pi}'_p$ has classical L-parameter ϕ. If we consider the maximal ideal $\mathfrak{m}\subset\mathbb{T}^\Sigma$ corresponding to $(\Pi')^\Sigma$, where $\mathbb{T}^\Sigma$ is the Hecke algebra of $\mathbf{G}'$ away from Σ, then $\mathfrak{m}$ is non-Eisenstein in the usual sense. Moreover, after localizing at $\mathfrak{m}$ and restricting to the summand on which $G(K)$ acts via a supercuspidal representation, we obtain an isomorphism

$$\begin{aligned}\Theta_{\mathfrak{m},\mathrm{sc}}:\mathrm{R}\Gamma_c(G,b,\mathbf{1},\{\mu\})_{\mathrm{sc}}\otimes\iota_\ell\left|-\right|_{K_1}^{\frac{-\dim_{\mathbb{C}}(\mathbf{X})}{2}}[\dim_{\mathbb{C}}(\mathbf{X})]\otimes^{\mathrm{L}}_{J(K)}\mathcal{A}\left(\mathbf{G}'(F)\backslash\mathbf{G}'(\mathbf{A}_{F,f})/\mathcal{K}^p,\mathscr{L}_{\iota_\ell\xi}\right)_{\mathfrak{m}}\\ \xrightarrow{\sim}\mathrm{R}\Gamma_c\left(\mathcal{S}(\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G},\mathbf{X})_{\mathcal{K}^p},\mathcal{L}_{\iota_\ell\xi}\right)_{\mathfrak{m}}.\end{aligned}$$

The assertion then follows if we can prove that the right-hand side is concentrated in the middle degree $\dim_{\mathbb{C}}(\mathbf{X})$ and carries a W_{K_1}-action given by

$$\widehat{\mathrm{Std}}\circ\phi\otimes\iota_\ell\left|-\right|_{K_1}^{\frac{-\dim_{\mathbb{C}}(\mathbf{X})}{2}}.$$

To prove this, we apply the Langlands–Kottwitz method in §4.5 to compute traces of sufficiently large powers of Frobenius at all but finitely many finite places of F. Information at the place p is then obtained from local–global compatibility for the Galois representation attached to the functorial transfer of $\dot{\Pi}'$, which is a self-dual cuspidal automorphic representation of $\mathrm{GL}_{2\,\mathrm{rank}(\mathbf{G}_{\overline{\mathbb{Q}}})}(\mathbf{A}_F)$.

Remark. A natural question is whether the same method can be applied to prove compatibility for other reductive groups, for example $\mathrm{GSpin}_n,\mathrm{GSp}_{2n},\mathrm{Sp}_{2n}$ and G_2. For inner forms of GSp_4 and Sp_4, this is known by [Ham25]. For GSpin_n, it is possible to extend the method to prove compatibility of Fargues–Scholze's construction with the correspondence constructed by Mœglin [Moe14] in the quasisplit case, once the endoscopic character identities, as formulated in [Kal16], are proved for all of their inner twists. On the other hand, new ideas are needed to treat the cases of GSp_{2n} ($n\geq 3$) and G_2, because a crucial step of the proof is to use the compatibility between local and global Shimura varieties to connect the construction of Fargues and Scholze with the so-called classical local Langlands correspondence through the cohomology of global Shimura varieties. The latter is studied via the Langlands–Kottwitz method, which can only give information about the (conjectural) global Galois representation $\rho:\mathrm{Gal}_F\to{}^LG$ associated with cohomological automorphic forms after composition with the extended highest weight module ${}^L\mathcal{T}_{\{\mu\}}$ of ${}^LG$, where $\{\mu\}$ is the conjugacy class of Hodge cocharacters of the Shimura datum. However, G_2 admits no Shimura variety, and in the case of $G=\mathrm{GSp}_{2n}$, the extended highest weight module $\mathcal{T}_{\{\mu\}}$ is the spin representation of $\mathrm{GSpin}_{2n+1}(\mathbb{C})$, so it is hard to recover the Galois representation and its local components. See [KS23] for related study of the cohomology of Siegel modular varieties.

In §§2.1–2.3, we review the classical local Langlands correspondence for special orthogonal and unitary groups and the statement of the endoscopic character identities. In §§2.4–2.6, we analyze more properties of the local Langlands correspondence. In §§3.1–3.3, we review the Fargues–Scholze local Langlands correspondence and the spectral action, and recall the related objects. In §3.4, we prove a weaker version of the Kottwitz conjecture. In §§4.1–4.3, we review the endoscopic classification of automorphic representations of relevant groups, and define a class of cohomological cuspidal automorphic representations with local constraints. In §4.5, we apply the Langlands–Kottwitz method to compute the Galois cohomology of relevant global Shimura varieties. In §5, we apply basic uniformization and Boyer's trick to prove a key property of the cohomology of relevant local Shimura varieties, see Corollary 5.2.3. In §6, we combine the previous results to prove the compatibility theorem, Theorem A. In §7.1, we use the compatibility property to construct an unambiguous local Langlands correspondence for even orthogonal groups. In §7.2, we prove the naturality property of the Fargues–Scholze local Langlands correspondence. In §7.3, we prove Theorem C by combining the compatibility result with the spectral actions. In §8.1, we study certain properties of generic toral L-parameters. In §8.2, we use the naturality of Fargues–Scholze local Langlands correspondence to prove the torsion vanishing result for Shimura varieties of orthogonal or unitary type. In §A, we review the endoscopy theory used in the main body.

1.3. **Notation and conventions.** We fix the following general notation.

Notation 1.3.1.

- Let $\mathbb{Z}_+$ denote the set of positive integers and $\mathbb{N}$ denote the set of non-negative integers.
- For each $n \in \mathbb{Z}_+$, we define $[n]_+ := \{1, 2, \dots, n\}$. For each $n \in \mathbb{N}$, we define $[n] := \{0, 1, \dots, n\}$.
- For each $n \in \mathbb{Z}_+$, let Sym_n denote the symmetric group on n letters.
- Suppose X is a set.
  - Let $\#X$ denote the cardinality of X and let $\mathscr{P}(X)$ denote the power set of X.
  - If X has a distinguished trivial element, we denote it by $\mathbf{1}$.
  - For two elements a, b in a set X, we define the Kronecker symbol
$$\delta_{a,b} := \begin{cases} 1 & \text{if } a = b \\ 0 & \text{if } a \neq b \end{cases}.$$
- Let $\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$ denote the fields of rational, real, and complex numbers, respectively. We fix a choice of square root i of -1 in $\mathbb{C}$.
- When A is a (topological/algebraic) group, we write $B \leq A$ to mean that B is a (closed) subgroup of A.
- For a finite group A, let $\mathrm{Irr}(A)$ denote the set of isomorphism classes of irreducible complex representations of A.
- All rings are assumed to be commutative and unital, and ring homomorphisms preserve units. Algebras, however, may be non-commutative and non-unital.
- The transpose of a matrix M is denoted by $M^\top$. When M is invertible, we write $M^{-\top}$ for $(M^{-1})^\top$.
- Let $J_n = (a_{ij})$ denote the anti-diagonal $n \times n$ matrix such that $a_{i,j} = \delta_{i,n+1-j}$ and $J'_n = (b_{ij})$ denote the anti-diagonal $n \times n$ matrix such that $b_{i,j} = (-1)^{i+1}\delta_{i,n+1-j}$.
- If S is a scheme over a commutative ring R and R' is a ring over R, we define $S_{R'} := S \times_{\operatorname{Spec} R} \operatorname{Spec} R'$.
- For a locally algebraic group scheme G over a field K, let $Z(G)$ denote the center of G and G° denote the identity component of G.

- Reductive groups are assumed to be connected.
- For a reductive group G over a field K, let W_G denote the relative Weyl group and G^* denote the unique quasisplit inner form of G. A Borel pair for G^* is defined to be a pair (B^*, T^*) consisting of a Borel subgroup B^* and a maximal torus T^* contained in B^*.

We fix the following notation for a connected reductive group over a non-Archimedean local field of characteristic zero.

Notation 1.3.2. Suppose $K/\mathbb{Q}_p$ is a finite extension and G is a connected reductive group over K.

- Let κ denote the residue field of K with a fixed algebraic closure $\overline{\kappa}$, and we fix a uniformizer $\varpi_K \in K^\times$.
- Let $\mathrm{ord}_K : K^\times \to \mathbb{Z}$ denote the additive valuation map that sends a uniformizer ϖ_K to 1, and let $|-|_K : K^\times \to \mathbb{R}_+$ denote the multiplicative valuation map such that $|x|_K = (\#\kappa)^{-\mathrm{ord}_K(x)}$.
- We fix an algebraic closure $\overline{K}$ of K, and for each subfield $K' \subset \overline{K}$, we define $\mathrm{Gal}_{K'} := \mathrm{Gal}(\overline{K}/K')$.
- Denote by W_K the Weil group of K and by I_K the inertia group of K. Let $\mathrm{Art}_K : K^\times \to W_K^{\mathrm{ab}}$ denote the Artin map. Fix an arithmetic Frobenius element $\varphi_K \in W_K$. Set $\sigma_K := \varphi_K^{-1}$, and we use the same symbol $|-|_K$ to denote the composition $W_K \to W_K^{\mathrm{ab}} \xrightarrow{\mathrm{Art}_K^{-1}} K^\times \xrightarrow{|-|_K} \mathbb{R}_+$.
- Let $\breve{K}$ denote the completion of the maximal unramified extension of K.
- We use the geometric normalization of the local class field theory, i.e., Artin maps are normalized so that they map uniformizers to geometric Frobenius classes.
- Let $\widehat{\mathsf{G}}$ denote the Langlands dual group of G, which is a Chevalley group whose based root data is dual to that of G. It is equipped with an action of Gal_K. Denote by ${}^L\mathsf{G} := \widehat{\mathsf{G}} \rtimes W_K$ the Langlands L-group of G in the Weil form. We usually conflate ${}^L\mathsf{G}$ (respectively, $\widehat{\mathsf{G}}$) with their $\mathbb{C}$-valued points, unless we write ${}^L\mathsf{G}(\Lambda)$ (respectively, $\widehat{\mathsf{G}}(\Lambda)$), which denotes its Λ-valued points for some ring Λ.
- Let $\mathcal{H}(\mathsf{G})$ denote the set of compactly supported locally constant $\mathbb{C}$-valued functions on $\mathsf{G}(K)$ that are bi-$\mathcal{K}$-finite for some compact open subgroup $\mathcal{K} \leq \mathsf{G}(K)$.
- If G, G' are reductive groups over K and π, π' are irreducible admissible representations of $\mathsf{G}(K)$ and $\mathsf{G}'(K)$, respectively, let $\pi \boxtimes \pi'$ denote the irreducible admissible representation of $\mathsf{G}(K) \times \mathsf{G}'(K)$ such that $(\pi \boxtimes \pi')((g, g')) = \pi(g) \otimes \pi'(g')$.
- Let $\Pi(\mathsf{G})$ denote the set of isomorphism classes of irreducible admissible representations of $\mathsf{G}(K)$, and let $\Pi_{\mathrm{temp}}(\mathsf{G})$ (resp. $\Pi_2(\mathsf{G})$, resp. $\Pi_{\mathrm{sc}}(\mathsf{G})$) denote the subset of $\Pi(\mathsf{G})$ consisting of tempered (resp. essentially square-integrable, resp. supercuspidal) representations. Set $\Pi_{2,\mathrm{temp}}(\mathsf{G}) := \Pi_2(\mathsf{G}) \cap \Pi_{\mathrm{temp}}(\mathsf{G})$.
- If $\mathsf{P} \leq \mathsf{G}$ is a parabolic subgroup with a Levi factor M and $\sigma \in \Pi(\mathsf{M})$, we let $\delta_{\mathsf{P}} : \mathsf{P}(K) \to \mathbb{R}_+$ denote the modulus quasi-character. We define the normalized parabolic induction by
$$\mathrm{I}_{\mathsf{P}}^{\mathsf{G}}(\sigma) := \mathrm{Ind}_{\mathsf{P}(K)}^{\mathsf{G}(K)} \left(\delta_{\mathsf{P}}^{1/2} \otimes \sigma\right).$$
- If $\mathsf{G} = \mathrm{GL}_n$ is a general linear group, we define a character $\nu := |-|_K \circ \det : \mathsf{G}(K) \to \mathbb{R}_+$.
- Suppose ℓ is a rational prime different from p and $\Lambda \in \{\overline{\mathbb{Q}_\ell}, \overline{\mathbb{F}_\ell}\}$. Let $\mathrm{D}(\mathsf{G}, \Lambda)$ denote the derived category of smooth representations of $\mathsf{G}(K)$ with coefficients in Λ, equipped with the natural t-structure. Let $\mathrm{D}^{\mathrm{adm}}(\mathsf{G}, \Lambda)$ denote the full subcategory of admissible complexes, i.e., those complexes whose invariants under any compact open subgroup $\mathcal{K} \leq \mathsf{G}(K)$ form a perfect complex.

- Suppose ℓ is a rational prime different from p and $\Lambda \in \{\overline{\mathbb{Q}_\ell}, \overline{\mathbb{F}_\ell}\}$. For each conjugacy class of cocharacters $\{\mu\}$ for $\mathsf{G}_{\overline{K}}$, there exists an indecomposable highest weight tilting module $\mathcal{T}_{\{\mu\}} \in \mathrm{Rep}_\Lambda(\widehat{\mathsf{G}})$ as defined in [Rin91, Don93]; cf. [Ham24, §9.1].
- We define
$$X_*(\mathsf{G}) := \mathrm{Hom}_K(\mathrm{GL}_{1,K}, \mathsf{G}), \quad X_\bullet(\mathsf{G}) := \mathrm{Hom}_{\overline{K}}(\mathrm{GL}_{1,\overline{K}}, \mathsf{G}_{\overline{K}})$$
for the set of cocharacters and geometric cocharacters of G, respectively, and define
$$X^*(\mathsf{G}) := \mathrm{Hom}_K(\mathsf{G}, \mathrm{GL}_{1,K}), \quad X^\bullet(\mathsf{G}) := \mathrm{Hom}_{\overline{K}}(\mathsf{G}_{\overline{K}}, \mathrm{GL}_{1,\overline{K}})$$
for the set of characters and geometric characters of G, respectively.
- For any condensed ∞-category $\mathcal{C}$ and any finite index set I, let $\mathcal{C}^{\mathrm{BW}_K^I}$ denote the category of objects with continuous W_K^I-actions, as defined in [FS24, §IX.1].
- For any subfield $\kappa' \subset \overline{\kappa}$, let $\mathcal{P}\mathrm{erfd}_{\kappa'}$ denote the category of affinoid perfectoid spaces over κ'.
- The six functor formalism of [Sch22] and [FS24] on ℓ-adic cohomology of diamonds and small Artin v-stacks is freely used. In particular, if ℓ is a rational prime different from p and $\Lambda \in \{\overline{\mathbb{Q}_\ell}, \overline{\mathbb{F}_\ell}\}$, then for any small Artin v-stack X, let $\mathrm{D}_{\blacksquare}(X, \Lambda)$ denote the condensed ∞-category of solid Λ-sheaves on X [FS24, §VII.1], and let $\mathrm{D}_{\mathrm{lis}}(X, \Lambda) \subset \mathrm{D}_{\blacksquare}(X, \Lambda)$ denote the full subcategory of Λ-lisse-étale sheaves as defined in [FS24, §VII.6].

We fix the following notation for a connected reductive group over a number field F.

Notation 1.3.3. Suppose F is a number field with a fixed embedding $\tau_0 : F \to \mathbb{C}$ and $\mathbb{G}$ is an arbitrary connected reductive group over F.

- We denote by Σ_F^{fin} the set of finite places of F, by Σ_F^∞ the set of infinite places of F. Set $\Sigma_F := \Sigma_F^{\mathrm{fin}} \cup \Sigma_F^\infty$.
- For each finite set S of rational primes, let $\Sigma_F(S) \subset \Sigma_F^{\mathrm{fin}}$ denote the subset of all finite places of F with residue characteristic in S.
- Let $\overline{F}$ denote the algebraic closure of F in $\mathbb{C}$.
- For each finite place v of F, let κ_v denote the residue field of F_v. Fix a decomposition group $D_v \subset \mathrm{Gal}_F$ and choose an element $\sigma_v \in D_v$ whose image in $\mathrm{Gal}(\overline{\kappa}_v/\kappa_v)$ is geometric Frobenius. We also set $\|v\| := \#\kappa_v$.
- Let $\mathbf{A}_F$ denote the ring of adeles of F, and let $\mathbf{A}_{F,f}$ denote the ring of finite adeles of F. We also write $\mathbf{A} := \mathbf{A}_\mathbb{Q}$ and $\mathbf{A}_f := \mathbf{A}_{\mathbb{Q},f}$.
- If $\Sigma \subset \Sigma_F^{\mathrm{fin}}$ is a finite subset, set $\mathbf{A}_{F,f}^\Sigma := \prod'_{v \in \Sigma_F^{\mathrm{fin}} \smallsetminus \Sigma} F_v$.
- For each discrete automorphic representation Π of $\mathbb{G}(\mathbf{A}_F)$, let $m(\Pi)$ denote its multiplicity in the discrete automorphic spectrum of $\mathbb{G}$.

2. Local Langlands correspondence via endoscopy

We begin by recalling the local Langlands correspondence defined via the theory of endoscopy. Let K be a non-Archimedean local field of characteristic zero, and fix a nontrivial additive character ψ_K of K, which extends to an additive character of any finite extension K'/K by defining $\psi_{K'} := \psi_K \circ \mathrm{tr}_{K'/K}$.

2.1. **The groups.** Let K_1/K be an unramified extension of degree at most two, and let $\mathsf{c} \in \mathrm{Gal}(K_1/K)$ be the element with fixed field K. Let $\chi_{K_1/K} : K^\times \to \{\pm 1\}$ denote the character associated with K_1/K via local class field theory. Let V be a vector space of dimension over K_1 equipped with a nondegenerate Hermitian c-sesquilinear form $\langle -, - \rangle$ on V, that is,
$$\langle au + bv, w \rangle = a \langle u, w \rangle + b \langle v, w \rangle, \quad \text{and} \quad \langle v, w \rangle = \langle w, v \rangle^{\mathsf{c}}$$

for all $a, b \in K_1$ and $u, v, w \in V$.

Fix an arbitrary orthogonal basis $\{v_1, \ldots, v_n\}$ of V such that $\langle v_i, v_i \rangle = a_i \in K^\times$. Define the discriminant of V as

$$\operatorname{disc}(V) = (-1)^{\binom{n}{2}} \prod_{i=1}^{\dim(V)} a_i.$$

The class of $\operatorname{disc}(V)$ in $K^\times/(K^\times)^2$ (resp. in $K^\times/\operatorname{Nm}_{K_1/K}(K_1^\times)$) when $K_1 = K$ (resp. when $K_1 \neq K$) is independent of the choice of orthogonal basis.

The (normalized) Hasse–Witt invariant of V is defined as

$$\epsilon(V) = \begin{cases} \left(-1, (-1)^{\binom{\dim(V)}{4}} \cdot \operatorname{disc}(V)^{\binom{\dim(V)-1}{2}}\right)_K \cdot \prod_{i<j \in [\dim(V)]_+} (a_i, a_j)_K & \text{if } K_1 = K, \\ \chi_{K_1/K}(\operatorname{disc}(V)) & \text{if } K_1 \neq K. \end{cases}$$

where

$$(-,-)_K : \left(K^\times/(K^\times)^2\right) \times \left(K^\times/(K^\times)^2\right) \to \operatorname{Br}(K)[2] \cong \{\pm 1\}$$

denotes the Hilbert symbol.

Recall from [Ser73, Theorem 2.3.7] that if $K_1 = K$, the isometry class of $(V, \langle -,- \rangle)$ is fully determined by the triple

$$(\dim V, \operatorname{disc}(V), \epsilon(V)) \in \mathbb{Z}_+ \times \left(K^\times/(K^\times)^2\right) \times \{\pm 1\},$$

and, moreover, it follows from [Ser73, Proposition 2.3.6] that all such triples except $(1, d, -1)$ and $(2, 1, -1)$ occur. If $K_1 \neq K$, the isometry class of $(V, \langle -,- \rangle)$ is completely characterized by the pair

$$(\dim(V), \epsilon(V)) \in \mathbb{Z}_+ \times \{\pm 1\},$$

and all such pairs occur; see [MH73].

Let $G(V)$ denote the algebraic subgroup of $\operatorname{GL}(V)$ such that

$$G(V) = \{g \in \operatorname{GL}(V) : \langle gv, gw \rangle = \langle v, w \rangle \, \forall v, w \in V\},$$

and let $G = G(V)^\circ$ denote its identity component. Let G^* denote the unique quasisplit inner form of G over K. Exactly one of the following three cases holds:

O1 $K_1 = K$ and $\dim(V) = 2n+1$ is odd. Then $G^* = \mathrm{SO}_{2n+1}$, the split orthogonal group in $2n+1$ variables.

O2 $K_1 = K$ and $\dim(V) = 2n$ is even. Then $G^* = \mathrm{SO}_{2n}^{\operatorname{disc}(V)}$, the quasisplit special orthogonal group associated with the quadratic space V^* over K of dimension $2n$, discriminant $\operatorname{disc}(V)$ and Hasse–Witt invariant 1.

U $K_1 \neq K$ and $\dim(V) = n$. Then $G^* = \mathrm{U}(n)$, the quasisplit unitary group associated with the Hermitian space of dimension n with respect to the unramified quadratic extension K_1/K, with Hasse–Witt invariant 1.

We collectively refer to Cases O1 and O2 together as Case O. Note that

- In Case O1, G is split if $\epsilon(V) = 1$ and non-quasisplit if $\epsilon(V) = -1$. In either case, G splits over the unramified quadratic extension of K.
- In Case O2, G is split if $\operatorname{disc}(V) = 1, \epsilon(V) = 1$; non-quasisplit if $\operatorname{disc}(V) = 1, \epsilon(V) = -1$; and quasisplit but non-split if $\operatorname{disc}(V) \neq 1$. Moreover, G splits over the unramified quadratic extension of K if and only if $\operatorname{ord}_K(\operatorname{disc}(V)) \equiv 0 (\mathrm{mod}\ 2)$.
- In Case U, G is non-quasisplit if n is even and $\epsilon(V) = -1$; otherwise it is quasisplit but non-split. In all cases, G splits over K_1.

To unify notation, let $n(G) = n(G^*)$ denote the geometric rank of G. Thus $n(\mathrm{SO}_{2n+1}) = n(\mathrm{SO}_{2n}^{\mathrm{disc}(V)}) = n(\mathrm{U}_n) = n$. We define the following invariants associated with G:

$$N(G) := \begin{cases} 2n(G) & \text{in Case O,} \\ n(G) & \text{in Case U,} \end{cases}$$
$$d(G) := \begin{cases} 2n(G)+1 & \text{in Case O1,} \\ 2n(G) & \text{in Case O2,} \\ n(G) & \text{in Case U,} \end{cases} \tag{2.1}$$
$$b(G) := \begin{cases} -1 & \text{in Case O1,} \\ 1 & \text{in Case O2,} \\ (-1)^{n(G)-1} & \text{in Case U.} \end{cases}$$

Here $N(G)$ is the integer N for the group GL_N associated with the Langlands dual group of G; $d(G)$ is the dimension of the c-Hermitian space V defining G, and $b(G)$ is the sign associated with G. In Case O2, we also define $\mathrm{disc}(G) := \mathrm{disc}(V)$.

Let $\mathcal{F} = \{0 = X_0' \subset X_1' \subset X_2' \subset \cdots \subset X_r'\}$ be a flag of isotropic K_1-subspaces of V. Then there exists an orthogonal direct sum decomposition

$$V = (X_r' \oplus Y_r') \perp V',$$

where Y_r' is an isotropic subspace. The stabilizer $P \leq G$ of this flag $\mathcal{F}$ is a parabolic subgroup, and every parabolic subgroup of G arises in this way. Moreover, if X_i is a complement of X_{i-1}' in X_i' for each $i \in [r]_+$, then

$$M = \mathrm{Res}_{K_1/K}\mathrm{GL}(X_1) \times \cdots \times \mathrm{Res}_{K_1/K}\mathrm{GL}(X_r) \times G(V')^\circ$$

is a Levi subgroup of G (here $G(V')^\circ$ is trivial when $\dim V' = 0$). Every Levi subgroup of G arises in this way, and any two such Levi subgroups that are isomorphic are conjugate under $G(V)$.

We fix a pinning $(B^*, T^*, \{X_\alpha^*\}_{\alpha\in\Delta})$ of G^* by identifying it with $G(V^*)^\circ$ for a suitable c-Hermitian space V^* over K_1, and choosing a complete flag of totally isotropic subspaces in V^*. Recall that a Whittaker datum for G^* is a $T^*(K)$-conjugacy class of generic characters of $N^*(K)$, where N^* is the unipotent radical of B^*. Whittaker data for G^* form a principal homogeneous space over the finite Abelian group

$$E = \mathrm{Coker}\,(G^*(K) \to G^*_{\mathrm{ad}}(K)) = \ker(H^1(K, Z(G^*)) \to H^1(K, G^*));$$

see [GGP12, §9]. The fixed pinning of G^*, together with the additive character ψ_K of K, determines a Whittaker datum $\mathfrak{w}$ for G^*; see [KS99, §5.3]. When G is unramified, there exists a unique $G(K)$-conjugacy class of hyperspecial maximal compact open subgroups compatible with $\mathfrak{w}$, in the sense of [CS80]. In this case, "unramified representations of $G(K)$" refers to those unramified with respect to such a hyperspecial subgroup.

We define the Witt tower associated with G: For each $n_0 \in [n(G)]$, let $G(n_0)$ denote the reductive group (possibly the trivial group $\mathbf{1}$) of geometric rank n_0, such that there exists a Levi subgroup of G isomorphic to

$$\mathrm{Res}_{K_1/K}\mathrm{GL}_{\frac{n(G)-n_0}{[K_1:K]}} \times G(n_0).$$

By [Tit79, §4.4],

- In Case O1, $G(n_0)$ exists if and only if $n_0 \geq \frac{1-\epsilon(V)}{2}$,
- In Case O2, $G(n_0)$ exists if and only if $n_0 \geq 1 - \delta_{\mathrm{disc}(V),1} \cdot \epsilon(V)$,
- In Case U, $G(n_0)$ exists if and only if $n(G) - n_0$ is even and moreover $n_0 \neq 0$ when G is non-quasisplit (i.e., when $n(G)$ is even and $\epsilon(V) = -1$).

We fix an isomorphism

$$\widehat{G} \cong \begin{cases} \mathrm{Sp}_{N(G)}(\mathbb{C}) & \text{in Case O1} \\ \mathrm{SO}_{N(G)}(\mathbb{C}) & \text{in Case O2} \\ \mathrm{GL}_{N(G)}(\mathbb{C}) & \text{in Case U} \end{cases},$$

and fix a pinning $(\widehat{T}, \widehat{B}, \{X_\alpha\})$ where $\widehat{T}$ is the diagonal torus, $\widehat{B}$ is the group of upper triangular matrices, and $\{X_\alpha\}$ is the set of standard root vectors. Let ${}^L G = \widehat{G} \rtimes W_K$ denote the Langlands L-group in the Weil form, where W_K acts on $\widehat{G}$ preserving the pinning, with the action as follows:

- In Case O1, W_K acts trivially on $\widehat{G}$.
- In Case O2, W_K acts via the quotient $\mathrm{Gal}(K(\sqrt{\mathrm{disc}(G)})/K)$. If $\mathrm{disc}(G) \neq 1$ and $\widehat{G}$ is identified with the subgroup of $\mathrm{SL}(N(G), \mathbb{C})$ preserving the nondegenerate bilinear form on $\mathbb{C}\{v_1, \ldots, v_{N(G)}\}$ defined by

$$\langle v_i, v_j \rangle = \delta_{i,N(G)+1-j},$$

then the nontrivial element acts by conjugation via the element in $\mathrm{O}(N(G), \mathbb{C})$ that exchanges $v_{n(G)}$ and $v_{n(G)+1}$ and fixes the others.
- In Case U, W_K acts via the quotient $\mathrm{Gal}(K_1/K)$, where $\mathsf{c} \in \mathrm{Gal}(K_1/K)$ acts by

$$g \mapsto J_n' g^{-\top} (J_n')^{-1}.$$

Finally, note that $\widehat{G}$ has a standard representation

$$\widehat{\mathrm{Std}} = \widehat{\mathrm{Std}}_G : \widehat{G} \to \mathrm{GL}_{N(G)}(\mathbb{C}).$$

Let

$$\kappa_G : B(G)_{\mathrm{bas}} \xrightarrow{\sim} X^\bullet(Z(\widehat{G})^{\mathrm{Gal}_K})$$

denote the Kottwitz map [Kot85, Proposition 5.6], which induces an identification of $H^1(K, G)$ with $X^\bullet\left(\pi_0\left(Z(\widehat{G})^{\mathrm{Gal}_K}\right)\right)$. There is a canonical isomorphism

$$\mathrm{H}^1(K, G) = B(G)_{\mathrm{bas}} \cong \mathbb{Z}/2,$$

except in the case $G = \mathrm{SO}_2^1 \cong \mathrm{GL}_1$, where $H^1(K, G) = 1$ while $B(G)_{\mathrm{bas}} \cong \mathbb{Z}$; see [GGP12, Lemma 2.1]. We exclude the case $G^* = \mathrm{SO}_2^1$ from now on. Except when $G^* \cong \mathrm{SO}_2^1$, every extended pure inner twist of G^* is canonically a pure inner twist. We henceforth exclude the case $G^* \cong \mathrm{SO}_2^1$.

For every $b \in B(G)_{\mathrm{bas}}$, we may associate an *extended pure inner twist* (G_b, ϱ_b, z_b) as defined in [Kot97, §3.3, 3.4], where

- $G_b(K) = \left\{ g \in G(\breve{K}) | \dot{b} \varphi_K(g) \dot{b}^{-1} = g \right\}$, where $\dot{b} \in G(\breve{K})$ maps to $b \in B(G)_{\mathrm{bas}}$,
- $\varrho_b : G \otimes_K \overline{K} \xrightarrow{\sim} G_b \otimes_K \overline{K}$ is an isomorphism over $\overline{K}$, and
- z_b is a 1-cocycle in $Z^1(W_K, G_{\mathrm{ad}})$ representing the class corresponding to b.

The pointed set $\mathrm{H}^1(K, G)$ and the extended pure inner twists of G correspond bijectively to forms V' of the space V with its sesquilinear form $(-,-)$; cf. [KMRT98, §29D and §29E]. Choose an extended pure inner twist $(G^*_{b_0}, \varrho_{b_0}, z_{b_0})$ associated with $b_0 \in B(G)_{\mathrm{bas}}$ such that

$$G \cong G^*_{b_0}.$$

For each $b \in B(G)_{\mathrm{bas}}$, the inner twists ϱ_{b_0} and ϱ_b canonically identify $\widehat{G_b}$, $\widehat{G}$, and $\widehat{G^*}$.

For every parabolic pair (M, P) of G, there exists a unique standard parabolic pair (M^*, P^*) of G^* corresponding to (M, P) under ϱ_{b_0}. This determines an equivalence class of extended pure inner twists of M^*, which we also denote by (ϱ_{b_0}, z_{b_0}).

For each $b_0 \in B(G^*)_{\mathrm{bas}}$, we write κ_{b_0} for the character of $\pi_0(Z(\widehat{G})^{\mathrm{Gal}_K})$ corresponding to b_0 under the Kottwitz map. Since $Z(\widehat{G})^{\mathrm{Gal}_K}$ is finite, the Kottwitz maps for G and G^* give isomorphisms

$$B(G)_{\mathrm{bas}} \cong X^\bullet(Z(\widehat{G})^{\mathrm{Gal}_K}) \cong B(G^*)_{\mathrm{bas}}, \tag{2.2}$$

which sends b to $b + b_0$; cf. [Kot85, p. 202]. In particular, $G_b \cong G^*_{b_0+b}$.

There is an automorphism θ of $G^{\mathrm{GL}} := \mathrm{Res}_{K_1/K}\mathrm{GL}_{N(G)}$ given by

$$\theta(g) = J'_{N(G)}\mathfrak{c}(g)^{-\top}(J'_{N(G)})^{-1}$$

for

$$g \in G^{\mathrm{GL}}(K) = \mathrm{GL}_{N(G)}(K_1).$$

We fix a standard Gal_{K_1}-invariant pinning $(B^{\mathrm{GL}}, T^{\mathrm{GL}}, \{X^{\mathrm{GL}}_\alpha\})$ of G^{GL} stabilized by θ.

The quasisplit group G^* determines an elliptic twisted endoscopic datum for $G^{\mathrm{GL}} \rtimes \theta$, hence a class in $\mathcal{E}_{\mathrm{ell}}(G^{\mathrm{GL}} \rtimes \theta)$; see [Mok15, pp. 3–4, 7] in Case U and [Art13, §1.2] in Case O. Denote the group $\mathrm{OAut}_{G^{\mathrm{GL}}}(G^*)$ from (A.1) by $\mathrm{OAut}_N(G^*)$. Then $\mathrm{OAut}_N(G^*)$ is trivial in Case O1 and Case U, and $\mathrm{OAut}_N(G^*) = \mathrm{O}_{2n(G^*)}(\mathbb{C})/\mathrm{SO}_{2n(G^*)}(\mathbb{C})$ in Case O2.

We recall the following list of isomorphism classes of elliptic endoscopic triples $\mathfrak{e} \in \mathcal{E}_{\mathrm{ell}}(G^*)$ from [Rog90, §4.6] and [Wal10], where we only describe $G^{\mathfrak{e}}$ and $\mathrm{OAut}(\mathfrak{e})$:

- In Case U, $G^{\mathfrak{e}} \cong \mathrm{U}_{K_1/K}(a) \times \mathrm{U}_{K_1/K}(b)$. Here $a, b \in [n(G)]$ satisfy $a + b = n(G)$, and $\mathrm{OAut}_{G^*}(\mathfrak{e})$ is trivial except when $a = b$, where there exists a unique nontrivial outer automorphism swapping the two factors of $\widehat{G^{\mathfrak{e}}} \cong \mathrm{GL}_a \times \mathrm{GL}_b$.
- In Case O1, $G^{\mathfrak{e}} \cong \mathrm{SO}_{2a+1} \times \mathrm{SO}_{2b+1}$. Here $a, b \in [n(G)]$ satisfy $a+b = n(G)$, and $\mathrm{OAut}_{G^*}(\mathfrak{e})$ is trivial except when $a = b$, in which case there exists a unique nontrivial outer automorphism swapping the two factors of $\widehat{G^{\mathfrak{e}}} \cong \mathrm{Sp}_{2a}(\mathbb{C}) \times \mathrm{Sp}_{2b}(\mathbb{C})$.
- In Case O2, $G^{\mathfrak{e}} \cong \mathrm{SO}^\beta_{2a} \times \mathrm{SO}^\gamma_{2b}$. Here $a, b \in [n(G)]$ satisfy $a + b = n(G), \beta\gamma = \mathrm{disc}(G)$, and moreover $\beta = 1$ if $a = 0$, $\gamma = 1$ if $b = 0$, and $(a, \beta) \neq (1, 1), (b, \gamma) \neq (1, 1)$. If $ab > 0$, there exists an outer automorphism acting by conjugation action of an element of $\mathrm{O}_{2a}(\mathbb{C}) \times \mathrm{O}_{2b}(\mathbb{C})$ on the two factors of $\widehat{G^{\mathfrak{e}}} \cong \mathrm{SO}_{2a}(\mathbb{C}) \times \mathrm{SO}_{2b}(\mathbb{C})$. There are no other nontrivial outer automorphisms except when $a = b$ and $\mathrm{disc}(G) = 1$, in which case there exists one swapping the two factors of $\widehat{G^{\mathfrak{e}}} \cong \mathrm{SO}_{2a}(\mathbb{C}) \times \mathrm{SO}_{2b}(\mathbb{C})$, and so $\mathrm{OAut}(\mathfrak{e}) \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ in this case.

For each $\mathfrak{e} \in \mathcal{E}_{\mathrm{ell}}(G^*)$, we fix a choice of ${}^L\xi^{\mathfrak{e}}$ as in [Wal10, §1.8], such that if $G^{\mathfrak{e}} = H_1 \times H_2$, then $\widehat{\mathrm{Std}}_G \circ {}^L\xi^{\mathfrak{e}}$ is conjugate to $(\widehat{\mathrm{Std}}_{H_1} \oplus \widehat{\mathrm{Std}}_{H_2}) \circ \iota$, where $\iota : {}^LG^{\mathfrak{e}} \hookrightarrow {}^LH_1 \times {}^LH_2$ is the natural inclusion.

Non-elliptic endoscopic triples $\mathfrak{e} \in \mathcal{E}(G^*)$ are described similarly, in which case $G^{\mathfrak{e}}$ is a product of groups of the same type as G with geometric rank smaller than $n(G)$ and restrictions of scalars of general linear groups, see for example [Ish24, §3.1.3] in Case O1.

2.2. **The L-parameters.** We recall the description of L-parameters for G and G^*, and explain their relation to conjugate self-dual representations of Weil groups. Let $\Phi(G)$ denote the set of L-parameters for G, in the sense of [Bor79, §8.2]. We describe this set explicitly below.

For each positive integer $m \in \mathbb{Z}_+$, an L-parameter ϕ for $\mathrm{GL}_{m,K'}$ over any finite extension K'/K may be regarded as an isomorphism class of m-dimensional representations of $W_{K'} \times \mathrm{SL}_2(\mathbb{C})$. Every such representation is isomorphic to a finite direct sum of representations of the form $\rho \boxtimes \mathrm{sp}_a$ where ρ is an irreducible smooth representation of $W_{K'}$ and sp_a is the unique irreducible algebraic representation of $\mathrm{SL}_2(\mathbb{C})$ of dimension a.

An L-parameter ϕ for GL_m over K_1 is called conjugate self-dual and irreducible if ϕ is irreducible and isomorphic to $\phi^\theta := (\phi^s)^\vee$ as representations, where $s \in W_K$ is a lift of $\mathfrak{c} \in W_K/W_{K_1} \cong \mathrm{Gal}(K_1/K)$ and ϕ^s is the conjugate action $\phi^s(g) = \phi(sgs^{-1})$. Following [GGP12, §3], we introduce the sign of a conjugate self-dual irreducible L-parameter ϕ: There exists an isomorphism $f : \phi \xrightarrow{\sim} \phi^\theta$ such that $(f^\vee)^s = b(\phi)f$ for some $b(\phi) \in \{\pm 1\}$. The value $b(\phi)$ is independent of the choice of f, and is called the sign of ϕ. If $\phi = \rho \boxtimes \mathrm{sp}_a$, where ρ is an irreducible representation of W_{K_1}, then

$$b(\phi) = b(\rho)(-1)^{a-1};$$

see [GGP12, Lemma 3.2], [KMSW14, §1.2.4].

Then it follows from [GGP12, Theorem 8.1] and [AG17, p. 365] that there exists a natural identification

$$\Phi(G)=\Phi(G^*)=\begin{cases}\{\text{admissible } \phi : W_{K_1}\times \mathrm{SL}_2(\mathbb{C})\to \mathrm{Sp}_{N(G)}(\mathbb{C})\}/\,\mathrm{Sp}_{N(G)}(\mathbb{C}) & \text{in Case O1,}\\ \left\{\begin{aligned}&\text{admissible } \phi : W_{K_1}\times \mathrm{SL}_2(\mathbb{C})\to \mathrm{O}_{2n}(\mathbb{C})\\ &\text{such that } \det(\phi)=(\mathrm{Art}_K^{-1}(-),\mathrm{disc}(V))_K\end{aligned}\right\}/\,\mathrm{SO}_{2n}(\mathbb{C}) & \text{in Case O2,}\\ \left\{\begin{aligned}&\text{admissible } \phi : W_{K_1}\times \mathrm{SL}_2(\mathbb{C})\to \mathrm{GL}_{N(G)}(\mathbb{C})\\ &\text{that is conjugate self-dual of sign } (-1)^{n(G)-1}\end{aligned}\right\}/\mathrm{GL}_{N(G)}(\mathbb{C}) & \text{in Case U.}\end{cases}$$

Here ϕ is called admissible if

- $\phi(\sigma_{K_1})$ is semisimple,
- $\phi|_{I_{K_1}}$ is smooth, and
- $\phi|_{\mathrm{SL}_2(\mathbb{C})}$ is algebraic.

Strictly speaking, in the definition of [Bor79, §8.2], an L-parameter for G is a homomorphism from $W_K\times \mathrm{SL}_2(\mathbb{C})$ to ${}^LG$ up to $\widehat{G}$-conjugacy. In the description above, we instead encode such a parameter by the corresponding homomorphism with source $W_{K_1}\times \mathrm{SL}_2(\mathbb{C})$. Thus, by a slight abuse of terminology, we will refer to this latter homomorphism as an L-parameter and denote it by ϕ. The associated parameter in the sense of [Bor79, §8.2] will be denoted by

$$\phi^\natural : W_K\times \mathrm{SL}_2(\mathbb{C})\longrightarrow {}^LG^*.$$

Then an L-parameter $\phi\in\Phi(G^*)$ is

- tempered (or bounded) if and only if $\phi(W_{K_1})$ is a relatively compact subset of the target.
- discrete if and only if $\mathrm{Im}(\phi)$ is not contained in the normalizer of a proper parabolic subgroup of $\widehat{G}^*$.
- semisimple if and only if it is trivial on the $\mathrm{SL}_2(\mathbb{C})$-factor.
- supercuspidal if and only if it is discrete and semisimple.

The subset of tempered (resp. discrete, resp. semisimple, resp. supercuspidal) L-parameters for G^* is denoted by $\Phi_{\mathrm{temp}}(G^*)$ (resp. $\Phi_2(G^*)$, resp. $\Phi^{\mathrm{ss}}(G^*)$, resp. $\Phi^{\mathrm{sc}}(G^*)$).

An L-parameter $\phi\in\Phi(G^*)$ can be regarded as an $N(G)$-dimensional conjugate self-dual representation ϕ^{GL} of $W_{K_1}\times \mathrm{SL}_2(\mathbb{C})$ via the standard representation $\widehat{\mathrm{Std}}_G$. ϕ is determined by ϕ^{GL} in Case O1 and Case U, but only determined up to $\mathrm{O}_{N(G)}(\mathbb{C})$-conjugation in Case O2; see [GGP12, Theorem 8.1]. We can write ϕ^{GL} as

$$\phi^{\mathrm{GL}}=\bigoplus_{i\in I_\phi^+} m_i\phi_i\oplus\bigoplus_{i\in I_\phi^-} 2m_i\phi_i\oplus\bigoplus_{i\in J_\phi} m_i(\phi_i\oplus\phi_i^\theta),$$

where m_i are positive integers, and $I_\phi^+, I_\phi^-, J_\phi$ index mutually inequivalent irreducible representations of $W_{K_1}\times \mathrm{SL}_2(\mathbb{C})$ such that

- for $i\in I_\phi^+$, ϕ_i is conjugate self-dual with sign $b(G)$.
- for $i\in I_\phi^-$, ϕ_i is conjugate self-dual with sign $-b(G)$.
- for $i\in J_\phi$, ϕ_i is not conjugate self-dual.

Then ϕ is discrete if and only if $m_i=1$ for $i\in I_\phi^+$, and $I_\phi^-=J_\phi=\varnothing$. Moreover, we call ϕ a *simple L-parameter* if it is discrete and $\#I_\phi^+=1$.

For any $\phi\in\Phi(G^*)$, define

$$S_\phi^\sharp := \prod_{i\in I_\phi^+}\mathrm{O}_{m_i}(\mathbb{C})\times\prod_{i\in I_\phi^-}\mathrm{Sp}_{2m_i}(\mathbb{C})\times\prod_{i\in J_\phi}\mathrm{GL}_{m_i}(\mathbb{C}).$$

This group formally represents the centralizer of ϕ in $\widehat{G} \rtimes \mathrm{Gal}_{K'/K}$, where K' is a minimal splitting field of G. Its formal component group is

$$\mathfrak{S}_\phi^\sharp := \pi_0(S_\phi^\sharp) \cong \bigoplus_{i \in I_\phi^+} (\mathbb{Z}/2)e_i.$$

For each $i \in I_\phi^+$, let $e_i^\vee \in \mathrm{Irr}(\mathfrak{S}_\phi^\sharp)$ be the character determined by

$$e_i^\vee(e_j) = (-1)^{\delta_{i,j}} \qquad (i, j \in I_\phi^+).$$

Writing tensor products of characters additively, we obtain

$$\mathrm{Irr}(\mathfrak{S}_\phi^\sharp) \cong \bigoplus_{i \in I_\phi^+} (\mathbb{Z}/2)e_i^\vee.$$

In Case O2, define

$$S_\phi := Z_{\widehat{G}}(\phi).$$

It is naturally a subgroup of $S_\phi^\sharp$ of index at most two. The homomorphism

$$\det_\phi : \mathfrak{S}_\phi^\sharp \to \mathbb{Z}/2, \qquad \sum_{i \in I_\phi^+} x_i e_i \mapsto \sum_{i \in I_\phi^+} x_i \dim(\phi_i)$$

has kernel

$$\mathfrak{S}_\phi := \ker(\det_\phi) = \pi_0(S_\phi).$$

To unify notation, in Case O1 and Case U, set $S_\phi := S_\phi^\sharp$ and $\mathfrak{S}_\phi := \mathfrak{S}_\phi^\sharp$.

We define the central element

$$z_\phi := \sum_{i \in I_\phi^+} m_i e_i \in \mathfrak{S}_\phi,$$

and define the reduced component group

$$\overline{\mathfrak{S}}_\phi := \mathfrak{S}_\phi / \langle z_\phi \rangle.$$

When $(G = G^*_{b_0}, \varrho_{b_0}, z_{b_0})$ is a pure inner form of G^*, an L-parameter for G^* is called (G, ϱ_{b_0})-relevant if every Levi subgroup ${}^L M$ of ${}^L G$ such that $\mathrm{Im}(\phi) \subset {}^L M$ is relevant, i.e., ${}^L M$ is a Levi component of a (G, ϱ_{b_0})-relevant parabolic subgroup of ${}^L G$; see [KMSW14, §0.4.2].

Let M be a standard Levi subgroup of G of the form

$$\mathrm{Res}_{K_1/K}\mathrm{GL}_{n_1} \times \cdots \times \mathrm{Res}_{K_1/K}\mathrm{GL}_{n_k} \times G(n_0),$$

where

$$[K_1 : K](n_1 + \cdots + n_k) + n_0 = n(G).$$

Define the K_1-group

$$M^{\mathrm{GL}} := \mathrm{Res}_{K_1/K}\left(\mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_k} \times \mathrm{GL}_{\frac{N(G)}{n(G)}n_0}\right).$$

Then $\Phi(M^{\mathrm{GL}})$ is canonically identified with the set of tuples $(\phi_1, \ldots, \phi_k, \phi_0)$, where

$$\phi_i \in \Phi(\mathrm{GL}_{n_i,K_1}) \quad \text{and} \quad \phi_0 \in \Phi(\mathrm{GL}_{\frac{N(G)}{n(G)}n_0,K_1})$$

Then the canonical map $\Phi(M) \to \Phi(G)$ fits into the commutative diagram

$$
\begin{array}{ccc}
\Phi(M) & \xrightarrow{(-)^{\mathrm{GL}}} & \Phi(M^{\mathrm{GL}}) \\
\downarrow & & \downarrow \\
\Phi(G) & \xrightarrow{(-)^{\mathrm{GL}}} & \Phi(G^{\mathrm{GL}}),
\end{array}
\tag{2.3}
$$

where the right vertical map is given by

$$
(\phi_1, \ldots, \phi_k, \phi_0) \mapsto \phi^{\mathrm{GL}} = \phi_1 \oplus \cdots \oplus \phi_k \oplus \phi_0 \oplus \phi_1^\theta \oplus \cdots \oplus \phi_k^\theta. \tag{2.4}
$$

2.3. **The correspondence.** We state the local Langlands correspondence for pure inner twist (G, ϱ, z) of G^*. In Case O1, it is established by Arthur [Art13] when G is quasisplit, and by Ishimoto [Ish24, Theorem 3.15] when G is not quasisplit. In Case U it is established by Mok [Mok15, Theorem 2.5.1, Theorem 3.2.1] when G is quasisplit, and by Kaletha, Minguez, Shin and White [KMSW14, Theorem 1.6.1] when G is not quasisplit. In Case U, more properties of this correspondence are established in [CZ21b].

In Case O2, only a weak version of the local Langlands correspondence is established. This is due to the intrinsic nature of the endoscopy method, because when G^* is regarded as a twisted endoscopic group of $\mathrm{GL}_{N(G)}$, ϕ can only be recovered from ϕ^{GL} up to $\mathrm{O}_{N(G)}(\mathbb{C})$-conjugation [GGP12, Theorem 8.1]. When G is quasisplit, the weak LLC is established by Arthur [Art13] (see also [AG17, Theorem 3.6]), and when G is not quasisplit, it is established by Chen and Zou [CZ21a, Theorem A.1].

To state the weak version in Case O2, for each pure inner twist $(G = G^*_{b_0}, \varrho_{b_0}, z_{b_0})$ of G^*, we introduce an equivalence relation $\sim_\varsigma$ on $\Pi(G)$. Note that there exists an outer automorphism ς of G^* which preserves $\mathfrak{w}$: in fact ς can be realized as an element of the corresponding orthogonal group of determinant -1; see [Taï19, p. 847]. Via ϱ_{b_0}, the element ς acts by a rational outer automorphism on G; see [Art13, Lemma 9.1.1]. For each $\pi \in \Pi(G)$, its ς-conjugate π^ς is defined by $\pi^\varsigma(h) = \pi(h^\varsigma)$, and the equivalence relation $\sim_\varsigma$ is defined by

$$
\pi \sim_\varsigma \pi^\varsigma.
$$

For each $\pi \in \Pi(G)$, we write $\tilde{\pi}$ for the image of π under the quotient map $\Pi(G) \to \tilde{\Pi}(G)$, where $\tilde{\Pi}(G) := \Pi(G)/\sim_\varsigma$. It is clear that the sets $\Pi_{\mathrm{temp}}(G), \Pi_2(G)$ and $\Pi_{\mathrm{sc}}(G)$ are preserved under this equivalence relation, so temperedness, discreteness, and supercuspidality are well-defined for equivalence classes $\tilde{\pi} \in \tilde{\Pi}(G)$.

Similarly, we define

$$
\tilde{\Phi}(G^*) = \{\text{admissible } \phi : W_{K_1} \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{O}_{N(G)}(\mathbb{C}) |\det(\phi) = (\mathrm{Art}_K^{-1}(-), \mathrm{disc}(V))_K\}/\mathrm{O}_{N(G)}(\mathbb{C}),
$$

together with a natural map $\Phi(G^*) \to \tilde{\Phi}(G^*)$. Note that if ϕ_1 and ϕ_2 are conjugate up to $\mathrm{O}_{N(G)}(\mathbb{C})$, then $\phi_1^{\mathrm{GL}} = \phi_2^{\mathrm{GL}}$. In particular, $\phi^{\mathrm{ss}}, \phi^{\mathrm{GL}}, I_\phi^+, I_\phi^-, J_\phi, S_\phi^\sharp, S_\phi, \mathfrak{S}_\phi, \overline{\mathfrak{S}}_\phi, z_\phi$ are well-defined functions for $\tilde{\phi} \in \tilde{\Phi}(G^*)$. We also set $\tilde{\Phi}(G) := \tilde{\Phi}(G^*)$.

For uniformity of notation, in Case O1 and Case U set $\varsigma = \mathrm{id}_G$ and $\tilde{\Pi}(G) = \Pi(G)$, $\tilde{\Phi}(G) = \Phi(G)$. Define a subspace $\tilde{\mathcal{H}}(G) \subset \mathcal{H}(G)$ of test functions on $G(K)$ as follows. In Case O2, following Arthur [Art13], let $\tilde{\mathcal{H}}(G)$ denote the subspace of $\mathcal{H}(G)$ consisting of ς-invariant functions on $G(K)$; so that irreducible smooth representations of $\tilde{\mathcal{H}}(G)$ correspond to $\mathrm{O}(V)$-conjugacy classes of irreducible admissible representations of $G(K)$. Similarly, for each $\mathfrak{e} \in \mathcal{E}_{\mathrm{ell}}(G)$, set

$$
\tilde{\mathcal{H}}(G^{\mathfrak{e}}) = \tilde{\mathcal{H}}(G_1) \otimes \tilde{\mathcal{H}}(G_2),
$$

since $G^{\mathfrak{e}}$ is a product of two (possibly trivial) even special orthogonal groups over K. In Case O1 and Case U, simply take $\tilde{\mathcal{H}}(G) = \mathcal{H}(G)$ and $\tilde{\mathcal{H}}(G^{\mathfrak{e}}) = \mathcal{H}(G^{\mathfrak{e}})$.

We can now reformulate the local Langlands correspondence as follows.

Theorem 2.3.1 ([Art13, Mok15, KMSW14, CZ21a, CZ21b, Ish24]). *If $(G = G^*_{b_0}, \varrho_{b_0}, z_{b_0})$ is a pure inner twist of G^*, then there exists a map*

$$\mathrm{rec}_G : \tilde{\Pi}(G) \to \tilde{\Phi}(G)$$

with finite fibers. For any $\tilde{\phi} \in \tilde{\Phi}(G)$, we write $\tilde{\Pi}_{\tilde{\phi}}(G)$ for $\mathrm{rec}_G^{-1}(\tilde{\phi})$, called the (ambiguous) L-packet for $\tilde{\phi}$. This map satisfies the following properties:

(1) If $\tilde{\phi} \in \tilde{\Phi}(G)$ is not relevant for G in the sense of [KMSW14, Definition 0.4.14], *then $\tilde{\Pi}_{\tilde{\phi}}(G) = \varnothing$.*

(2) For each $\tilde{\phi} \in \tilde{\Phi}(G)$ and $\tilde{\pi} \in \tilde{\Pi}_{\tilde{\phi}}(G)$, $\tilde{\pi}$ is tempered if and only if $\tilde{\phi}$ is tempered, and $\tilde{\pi}$ is a discrete series representation if and only if $\tilde{\phi}$ is discrete.

(3) rec_G only depends on G but not on ϱ_{b_0} and z_{b_0}. For the fixed Whittaker datum $\mathfrak{w}$ of G^* which induces a Whittaker datum for each standard Levi factor of G, there exists a canonical bijection

$$\iota_{\mathfrak{w},b_0} : \tilde{\Pi}_{\tilde{\phi}}(G) \cong \mathrm{Irr}(\mathfrak{S}_{\tilde{\phi}}; \kappa_{b_0})$$

for each $\tilde{\phi} \in \tilde{\Phi}(G)$, where $\mathrm{Irr}(\mathfrak{S}_{\tilde{\phi}}; \kappa_{b_0})$ is the set of characters η of $\mathfrak{S}_{\tilde{\phi}}$ such that $\eta(z_{\tilde{\phi}}) = \kappa_{b_0}(-\mathbf{1})$. We write $\tilde{\pi} = \tilde{\pi}_{\mathfrak{w},b_0}(\tilde{\phi}, \eta)$ if $\tilde{\pi} \in \tilde{\Pi}_{\tilde{\phi}}(G)$ corresponds to $\eta \in \mathrm{Irr}(\mathfrak{S}_{\tilde{\phi}})$ via $\iota_{\mathfrak{w},b_0}$.

(4) (Compatibility with Langlands quotient) For $\tilde{\phi} \in \tilde{\Phi}(G)$, suppose

$$\tilde{\phi}^{\mathrm{GL}} = \left(\phi_1 \otimes |-|_{K_1}^{s_1}\right) \oplus \cdots \oplus \left(\phi_r \otimes |-|_{K_1}^{s_r}\right) \oplus \tilde{\phi}_0^{\mathrm{GL}} \oplus \left(\phi_r^\theta \otimes |-|_{K_1}^{-s_r}\right) \oplus \cdots \oplus \left(\phi_1^\theta \otimes |-|_{K_1}^{-s_1}\right),$$

where

- *$\phi_i \in \Phi_{2,\mathrm{temp}}(\mathrm{GL}_{d_i,K_1})$ for each $i \in [r]_+$, where $d_i > 0$,*
- *$\tilde{\phi}_0 \in \tilde{\Phi}_{\mathrm{temp}}(G^*(n_0))$,*
- *$s_1 \geq s_2 \geq \cdots \geq s_r > 0$,*
- *$[K_1 : K](d_1 + \cdots + d_r) + n_0 = n(G)$.*

Let $\tau_i \in \Pi_{2,\mathrm{temp}}(\mathrm{GL}_{d_i,K_1})$ correspond to ϕ_i for each $i \in [r]_+$, then there is a canonical identification

$$\tilde{\Pi}_{\tilde{\phi}}(G) = \left\{ \bar{\mathrm{I}}_P^G \left((\tau_1 \otimes \nu^{s_1}) \boxtimes \cdots \boxtimes (\tau_r \otimes \nu^{s_r}) \boxtimes \tilde{\pi}_0 \right) \middle| \tilde{\pi}_0 \in \tilde{\Pi}_{\tilde{\phi}_0}(G(n_0)) \right\}.$$

Here P is a parabolic subgroup of G with a Levi factor

$$M \cong \mathrm{Res}_{K_1/K}\mathrm{GL}_{d_1} \times \cdots \times \mathrm{Res}_{K_1/K}\mathrm{GL}_{d_r} \times G(n_0),$$

such that $M = \varrho_{b_0}(M^*)$ where M^* is a standard Levi subgroup of G^*, and

$$\bar{\mathrm{I}}_P^G \left((\tau_1 \otimes \nu^{s_1}) \boxtimes \cdots \boxtimes (\tau_r \otimes \nu^{s_r}) \boxtimes \tilde{\pi}_0 \right)$$

is the unique irreducible quotient of

$$\mathrm{I}_P^G \left((\tau_1 \otimes \nu^{s_1}) \boxtimes \cdots \boxtimes (\tau_r \otimes \nu^{s_r}) \boxtimes \tilde{\pi}_0 \right),$$

whose ς-equivalence class is well-defined in $\tilde{\Pi}(G)$. Moreover, if $\mathfrak{w}_0$ is the induced Whittaker datum on M^*, there exists a natural identification $\mathfrak{S}_{\tilde{\phi}_0} \cong \mathfrak{S}_{\tilde{\phi}}$, and $\iota_{\mathfrak{w},b_0}(\tilde{\pi}) = \iota_{\mathfrak{w}_0,b_0}(\tilde{\pi}_0)$ if

$$\tilde{\pi} = \bar{\mathrm{I}}_P^G \left((\tau_1 \otimes \nu^{s_1}) \boxtimes \cdots \boxtimes (\tau_r \otimes \nu^{s_r}) \boxtimes \tilde{\pi}_0 \right).$$

(5) (Compatibility with standard γ-factors) Suppose $\tilde{\pi} \in \tilde{\Pi}(G)$ with $\tilde{\phi} := \mathrm{rec}_G(\tilde{\pi})$, then for any quasi-character χ of $K^\times$, viewed as a character of W_K and then restricted to W_{K_1},

$$\gamma(\tilde{\pi}, \chi, \psi_K; s) = \gamma(\tilde{\phi}^{\mathrm{GL}} \otimes \chi, \psi_{K_1}; s).$$

Here the left-hand side is the standard γ-factor defined by Lapid–Rallis using the doubling zeta integral [LR05] *but modified in* [GI14], *and the right-hand side is the γ-factor defined in* [Tat79].

(6) *(Compatibility with Plancherel measures) Suppose $\tilde{\pi} \in \tilde{\Pi}(G)$ with $\tilde{\phi} := \mathrm{rec}_G(\tilde{\pi})$, then for any $d \in \mathbb{Z}_+$ and any $\tau \in \Pi(\mathrm{GL}_{d,K_1})$ with L-parameter ϕ_τ,*

$$\begin{aligned}\mu_{\psi_K}(\tau \otimes \nu^s \boxtimes \tilde{\pi}) =& \gamma(\phi_\tau \otimes (\tilde{\phi}^{\mathrm{GL}})^\vee, \psi_{K_1}; s) \cdot \gamma(\phi_\tau^\vee \otimes \tilde{\phi}^{\mathrm{GL}}, \psi_{K_1}^{-1}, -s) \\ & \cdot \gamma(\mathrm{R}_{G^*} \circ \phi_\tau, \psi_K; 2s) \cdot \gamma(\mathrm{R}_{G^*} \circ \phi_\tau^\vee, \psi_K^{-1}; -2s).\end{aligned}$$

Here the left-hand side is the Plancherel measure defined in [GI14, §12] *(cf.* [GI16, §A.7]*), and in the right-hand side R_{G^*} is the representation*

$$\mathrm{R}_{G^*} = \begin{cases} \mathrm{Sym}^2 & \textit{in Case O1} \\ \wedge^2 & \textit{in Case O2} \\ \mathrm{As}^{(-1)^{N(G)}} & \textit{in Case U} \end{cases}$$

where As^+ and As^- are the two Asai representations of $\mathrm{Res}_{K_1/K}\mathrm{GL}_{N(G)}$.

(7) *(Local intertwining relations) Suppose $\tilde{\phi} \in \tilde{\Phi}(G)$ admits a decomposition*

$$\tilde{\phi}^{\mathrm{GL}} = \phi_\tau + \tilde{\phi}_0^{\mathrm{GL}} + \phi_\tau^\theta,$$

where $\tau \in \Pi_{2,\mathrm{temp}}(\mathrm{GL}_{d,K_1})$ has L-parameter $\phi_\tau \in \Phi(\mathrm{GL}_{d,K_1})$, and $\tilde{\phi}_0 \in \tilde{\Phi}_{\mathrm{temp}}(G(n(G) - [K_1 : K]d))$ is tempered. Let $P \leq G$ be a maximal parabolic subgroup with a Levi factor

$$M \cong \mathrm{Res}_{K_1/K}\mathrm{GL}_d \times G(n(G) - [K_1 : K]d).$$

Assume that $M = \varrho_{b_0}(M^*)$, where M^* is a standard Levi subgroup of G^*, and let $\mathfrak{w}_0$ be the induced Whittaker datum on M^*. Then, for each $\eta_0 \in \mathrm{Irr}(\mathfrak{S}_{\tilde{\phi}_0})$,

$$\mathrm{I}_P^G(\tau \boxtimes \tilde{\pi}_{\mathfrak{w}_0,b_0}(\tilde{\phi}_0, \eta_0)) = \bigoplus_\eta \tilde{\pi}_{\mathfrak{w},b_0}(\tilde{\phi}, \eta)$$

as a $\tilde{\mathcal{H}}(G)$-module, where η runs through characters of $\mathfrak{S}_{\tilde{\phi}}$ that restrict to η_0 under the natural embedding $\mathfrak{S}_{\tilde{\phi}_0} \hookrightarrow \mathfrak{S}_{\tilde{\phi}}$. Moreover, if ϕ_τ is conjugate self-dual of sign $b(G)$, let

$$\mathrm{R}_{\mathfrak{w}}(w, \tau \boxtimes \tilde{\pi}_{\mathfrak{w}_0,b_0}(\tilde{\phi}_0, \eta_0)) \in \mathrm{End}_{G(K)}\left(\mathrm{I}_P^G(\tau \boxtimes \tilde{\pi}_{\mathfrak{w}_0,b_0}(\tilde{\phi}_0, \eta_0))\right)$$

be the normalized intertwining operator defined in [CZ21a, §7.1] *in Case O2 and in* [CZ21b, §5.2] *in Case* U*, and in* [Ish24, §4] *in Case O1. Here w is the unique nontrivial element in the relative Weyl group for M. Then the restriction of $\mathrm{R}_{\mathfrak{w}}(w, \tau \boxtimes \tilde{\pi}_{\mathfrak{w}_0,b_0}(\tilde{\phi}_0, \eta_0))$ to $\tilde{\pi}_{\mathfrak{w},b_0}(\tilde{\phi}, \eta)$ is the scalar multiplication by*

$$\begin{cases} \eta(e_\tau) & \textit{in Case O} \\ \eta(e_\tau)\kappa_{b_0}^d(-\mathbf{1}) & \textit{in Case U} \end{cases}$$

where e_τ is the element of $\mathfrak{S}_{\tilde{\phi}}$ corresponding to ϕ_τ.

Remark 2.3.2.

(1) The independence of rec_G with respect to ϱ and z is established in Case O by [CZ21a, Remark 4.6(2)] and in Case U by the argument before [CZ21b, Theorem 2.5.5].

(2) The compatibility of rec_G with standard γ-factors and Plancherel measures can be used to characterize rec_G and show compatibility with LLC constructed via exceptional isomorphisms in low dimensions, via [GI16, Lemma A.6]. For example, in Case O2, when G is quasisplit, it is shown that the construction of Chen–Zou using theta correspondence is compatible with that defined by Arthur [CZ21a, Theorem 9.1], and in Case U, when G

is quasisplit, it is shown that the construction of Chen–Zou using theta correspondence is compatible with that defined by Mok [CZ21b, Theorem 7.1.1].

(3) The local intertwining relation can be used to characterize the finer structure of L-packets, i.e., $\iota_{\mathfrak{w},b_0}$. In Case O1, it follows from [Art13, Proposition 2.3.1 and Theorems 2.2.1, 2.2.4, 2.4.1 and 2.4.4] when G is quasisplit and follows from the corresponding propositions in [Ish24, §4] when G is not quasisplit. In Case U, it follows from [Mok15, Theorem 3.2.1, 3.4.3] when G is quasisplit and [CZ21b, Theorem 2.5.1] when G is not quasisplit. In Case O2, it follows from [Art13, Proposition 2.3.1 and Theorems 2.2.1, 2.2.4, 2.4.1 and 2.4.4] when G is quasisplit and follows from [CZ21a, Theorem A.1] when G is not quasisplit. For the details, see, for example, [Ato17, Theorem 2.2].

(4) Note that in Case U, the LLC stated in [Mok15] and [KMSW14] is compatible with the arithmetic normalization of the local class field theory instead of the geometric normalization of the local class field theory, i.e., the Artin map sends a uniformizer to an arithmetic Frobenius class rather than to a geometric Frobenius class. One can pass between the two normalizations by using the compatibility of local Langlands correspondence with contragredients [Kal13]; cf. [BMN23, Theorem 2.5].

We will always call this correspondence the *classical local Langlands correspondence (LLC)*, as opposed to the Fargues–Scholze local Langlands correspondence to be defined in §3.

For later use, we define what a local functorial transfer is from G to G^{GL}:

Definition 2.3.3. Let $\pi \in \Pi(G)$ have classical L-parameter $\tilde{\phi}$. The unique irreducible admissible representation $\pi^{\mathrm{GL}} \in \Pi(G^{\mathrm{GL}})$ whose classical L-parameter satisfies $\phi_{\pi^{\mathrm{GL}}} = \tilde{\phi}^{\mathrm{GL}}$ is called the *local functorial transfer* of π.

We also need certain endoscopic character identities for the local Langlands correspondence. In Case O1, they are established by Arthur [Art13, Theorem 2.2.1] when G is quasisplit, and by Ishimoto [Ish24, Theorem 3.15] when G is not quasisplit. In Case U, they are established by Mok [Mok15] when G is quasisplit, and by Kaletha, Minguez, Shin and White [KMSW14, Theorem 1.6.1] when G is not quasisplit. In Case O2, they are established by Arthur [Art13, Theorem 2.2.1] when G is quasisplit, and are established by [Pen25, Theorem A] when G is not quasisplit. We adopt the notation for endoscopy from §A.

Theorem 2.3.4. *Fix a pure inner twist $(G = G^*_{b_0}, \varrho_{b_0}, z_{b_0})$ of G^*.*

(1) Suppose $\mathfrak{e} \in \mathcal{E}_{\mathrm{ell}}(G)$ and $\phi^{\mathfrak{e}} \in \Phi_{2,\mathrm{temp}}(G^{\mathfrak{e}})$ with $\phi = {}^L\xi^{\mathfrak{e}} \circ \phi^{\mathfrak{e}}$ and $s^{\mathfrak{e}} \in S_\phi = \mathfrak{S}_\phi$. Then each $f \in \tilde{\mathcal{H}}(G)$ has a transfer $f^{G^{\mathfrak{e}}}$ contained in $\tilde{\mathcal{H}}(G^{\mathfrak{e}})$, and

$$\sum_{\tilde{\pi}^{\mathfrak{e}} \in \tilde{\Pi}_{\tilde{\phi}^{\mathfrak{e}}}(G^{\mathfrak{e}})} \mathrm{tr}\left(f^{G^{\mathfrak{e}}} | \tilde{\pi}^{\mathfrak{e}}\right) = e(G) \sum_{\tilde{\pi} \in \tilde{\Pi}_{\tilde{\phi}}(G)} \iota_{\mathfrak{w},b_0}(\tilde{\pi})(s^{\mathfrak{e}}) \mathrm{tr}\left(f | \tilde{\pi}\right).$$

Here $e(G)$ is the Kottwitz sign of G as defined in [Kot83], $\Delta[\mathfrak{w}, \mathfrak{e}, z_{b_0}]$ is the transfer factor normalized in §A.2, and $\tilde{\Pi}_{\tilde{\phi}^{\mathfrak{e}}}(G^{\mathfrak{e}})$ is defined to be the (ambiguous) L-packet associated with $\tilde{\phi}^{\mathfrak{e}}$ as before.[3]

In particular, by the Weyl integration formula and its stable analogue (cf. [HKW22, Appendix A.1, especially (A.1.2)]*),*[4] *the preceding distributional identity is equivalent to*

$$\sum_{h \in G^{\mathfrak{e}}(K)_{\mathrm{s.reg}}/G^{\mathfrak{e}}(\overline{K})\text{-conj}} \Delta[\mathfrak{w}, \mathfrak{e}, z_{b_0}](h, g) \mathrm{S}\Theta_{\tilde{\phi}^{\mathfrak{e}}}(h) = e(G) \sum_{\tilde{\pi} \in \tilde{\Pi}_{\tilde{\phi}}(G)} \iota_{\mathfrak{w},b_0}(\tilde{\pi})(s^{\mathfrak{e}}) \Theta_{\tilde{\pi}}(g)$$

[3]This L-packet is well defined because $G^{\mathfrak{e}}$ is a product of special orthogonal or unitary groups.

[4]See also [AK24, Lemma 2.11.2] for the same Weyl-integration formula argument.

for any strongly regular semisimple element $g \in G(K)_{\mathrm{s.reg}}$, where

$$\mathrm{S}\Theta_{\tilde{\phi}^{\mathfrak{e}}} := \sum_{\tilde{\pi} \in \tilde{\Pi}_{\tilde{\phi}^{\mathfrak{e}}}(G^{\mathfrak{e}})} \Theta_{\tilde{\pi}},$$

and $\Theta_{\tilde{\pi}}$ is the average of Θ_{π} where π runs through the preimage of $\tilde{\pi}$ under the map

$$\Pi(G^{\mathfrak{e}}) \to \tilde{\Pi}(G^{\mathfrak{e}}).$$

(2) For any tempered L-parameter $\tilde{\phi} \in \tilde{\Phi}_{\mathrm{temp}}(G^*)$, if $f^{\mathrm{GL}} \in \mathcal{H}(G^{\mathrm{GL}})$, then f^{GL} has a transfer f^* to G^* contained in $\tilde{\mathcal{H}}(G^*)$, and

$$\sum_{\tilde{\rho} \in \tilde{\Pi}_{\tilde{\phi}}(G^*)} \mathrm{tr}(f^*|\tilde{\rho}) = \mathrm{tr}_{\theta}(f^{\mathrm{GL}}|\pi_{\tilde{\phi}^{\mathrm{GL}}}).$$

Here the right-hand side is the θ-twisted trace, and $\pi_{\tilde{\phi}^{\mathrm{GL}}} \in \Pi(G^{\mathrm{GL}})$ is associated with the L-parameter $\tilde{\phi}^{\mathrm{GL}}$ via LLC, and the left-hand side is a stable distribution of f^*, i.e., it depends only on the stable orbital integrals of f^*.

Finally, we remark that the classical local Langlands correspondence for inner twists (or rather K-groups) of special orthogonal or unitary groups over $\mathbb{R}$ and (ordinary) endoscopic character identities are known in complete generality [ABV92, Vog93, She08, She10, Art13], where we replace irreducible admissible representations by smooth Fréchet representations of moderate growth whose associated Harish–Chandra modules are admissible, as introduced by Casselman [Cas89] and Wallach [Wal92]. Moreover, the twisted endoscopic character identities are known by Arthur [Art13]. The real case is analogous to the p-adic case, except that there are more inner twists. In fact, we will only use results concerning discrete series L-packets. For a modern exposition in the discrete series case, see [AK24].

2.4. **Compatibility with parabolic induction.** In this subsection, we recall the definition of extended cuspidal support of a discrete series representation of $G(K)$, and deduce a compatibility property of the classical semisimplified L-parameters with parabolic induction. In Case U, the assertion is established in [MHN24, Proposition 2.11], but our proof is slightly simpler than the proof given there.

First, we recall that it is a theorem of Bernstein and Zelevinsky [BZ77, Theorem 2.5, Theorem 2.9] that all irreducible smooth representations of $G(K)$ can be constructed by parabolic induction from supercuspidal representations:

Theorem 2.4.1. *For any irreducible smooth representation $\pi \in \Pi(G)$, there exists a unique pair (M, σ) up to conjugacy by $G(K)$, where M is a Levi component of some parabolic subgroup $P \leq G$ and $\sigma \in \Pi_{\mathrm{sc}}(M)$ is a supercuspidal representation, such that π is a subrepresentation of $\mathrm{I}_P^G(\sigma)$. Such pairs (M, σ) are called* cuspidal supports *of π.*

For a discrete series representation $\pi \in \Pi_{2,\mathrm{temp}}(G)$, we write $\pi^{\mathrm{GL}} \in \Pi(\mathrm{GL}_{N(G),K_1})$ for the representation corresponding to the L-parameter $\tilde{\phi}_{\pi}^{\mathrm{GL}}$, and write $\mathrm{SuppCusp}^+(\pi)$ for the cuspidal support of π^{GL}, called the *extended cuspidal support* of π.

Lemma 2.4.2. *If $\pi \in \Pi_{2,\mathrm{temp}}(G)$ is a discrete series representation, and if (M, π_M) is a cuspidal support of π, then $\mathrm{SuppCusp}^+(\pi_M) = \mathrm{SuppCusp}^+(\pi)$.*

Proof. If G is quasisplit, this is established in [Moe14, p. 309]. We note that by the endoscopic character identities Theorem 2.3.4, the classical L-packets are the same as the L-packets defined in [Moe14, §6.4]. For details, the reader is referred to [MHN24, Proposition 2.10].

When G is not quasisplit, in view of the endoscopic character identities Theorem 2.3.4, we can reduce the theorem to the quasisplit case using the alternative definition of $\mathrm{SuppCusp}^+(\pi)$ via

Jacquet modules [Moe14] and the fact that transfers are compatible with parabolic induction and the formation of Jacquet modules; see [She82, Proposition 3.4.2] and [Moe14, §2.6]. $\square$

Proposition 2.4.3. *If $P \leq G$ is a proper parabolic subgroup with Levi factor M, $\pi_M \in \Pi(M)$ with semisimplified L-parameter $\tilde{\phi}^{\mathrm{ss}}_{\pi_M} : W_{K_1} \to {}^L M$,*[5] *and $\pi \in \Pi(G)$ is a subquotient of the normalized parabolic induction $\mathrm{I}_P^G(\pi_M)$, then the semisimplified L-parameter $\tilde{\phi}^{\mathrm{ss}}_{\pi}$ is given by*

$$\tilde{\phi}^{\mathrm{ss}}_{\pi} : W_{K_1} \xrightarrow{\tilde{\phi}^{\mathrm{ss}}_{\pi_M}} {}^L M \to {}^L G.$$

Proof. First, the assertion for general linear groups follows from the compatibility of the local Langlands correspondence for general linear groups with cuspidal support, by a result of Bernstein and Zelevinsky; cf. [Rod82, Théorème 4]. We argue by induction on $n(G)$.

For nested Levi subgroups $M_1 \leq M_2$, write

$$\iota^L_{M_1,M_2} : {}^L M_1 \longrightarrow {}^L M_2$$

for the canonical L-homomorphism.

We first reduce to the case in which (M, π_M) is a cuspidal support of π. Let $(M', \pi_{M'})$ be a cuspidal support of π_M. Transitivity of normalized parabolic induction implies that $(M', \pi_{M'})$ is also a cuspidal support of π. By the induction hypothesis applied to the classical-group factor of M, together with the general linear case,

$$\tilde{\phi}^{\mathrm{ss}}_{\pi_M} = \iota^L_{M',M} \circ \tilde{\phi}^{\mathrm{ss}}_{\pi_{M'}}.$$

Consequently, once the assertion is proved for the cuspidal support $(M', \pi_{M'})$ of π, we obtain

$$\tilde{\phi}^{\mathrm{ss}}_{\pi} = \iota^L_{M',G} \circ \tilde{\phi}^{\mathrm{ss}}_{\pi_{M'}} = \iota^L_{M,G} \circ \tilde{\phi}^{\mathrm{ss}}_{\pi_M}.$$

We may therefore assume that (M, π_M) is a cuspidal support of π.

Suppose first that π is non-tempered. By the Langlands classification of irreducible representations in terms of irreducible tempered representations (see [Sil78] and [Kon03, Theorem 3.5]), there are a parabolic subgroup $P_L \leq G$ with Levi factor

$$M_L \cong \mathrm{Res}_{K_1/K}\mathrm{GL}_{d_1} \times \cdots \times \mathrm{Res}_{K_1/K}\mathrm{GL}_{d_r} \times G(n_0),$$

tempered representations $\tau_i \in \Pi_{\mathrm{temp}}(\mathrm{GL}_{d_i,K_1})$ and $\pi_0 \in \Pi_{\mathrm{temp}}(G(n_0))$, and real numbers $s_1 \geq \cdots \geq s_r > 0$, such that π is the unique irreducible quotient of

$$\mathrm{I}^G_{P_L}\left((\tau_1 \otimes \nu^{s_1}) \boxtimes \cdots \boxtimes (\tau_r \otimes \nu^{s_r}) \boxtimes \pi_0\right).$$

Compatibility of the classical correspondence with Langlands quotients, Theorem 2.3.1, gives

$$\begin{aligned}\tilde{\phi}^{\mathrm{GL}}_{\pi} = &\left(\phi_{\tau_1} \otimes |-|^{s_1}_{K_1}\right) \oplus \cdots \oplus \left(\phi_{\tau_r} \otimes |-|^{s_r}_{K_1}\right) \oplus \tilde{\phi}^{\mathrm{GL}}_{\pi_0}\\ &\oplus \left(\phi^{\theta}_{\tau_r} \otimes |-|^{-s_r}_{K_1}\right) \oplus \cdots \oplus \left(\phi^{\theta}_{\tau_1} \otimes |-|^{-s_1}_{K_1}\right).\end{aligned}$$

The general linear case and the induction hypothesis for $G(n_0)$, together with transitivity of normalized parabolic induction and uniqueness of cuspidal support, now give the desired formula for $\tilde{\phi}^{\mathrm{ss}}_{\pi}$.

We may henceforth assume that π is tempered. By the classification of tempered representations of classical groups [Jan14], there are a Levi subgroup $M_L \leq G$ and a discrete series representation $\sigma \in \Pi_{2,\mathrm{temp}}(M_L)$ such that π is an irreducible constituent of $\mathrm{I}^G_{P_L}(\sigma)$, and

$$\tilde{\phi}_{\pi} = \iota^L_{M_L,G} \circ \tilde{\phi}_{\sigma}.$$

[5]The L-parameter $\tilde{\phi}^{\mathrm{ss}}_{\pi_M}$ is well-defined because M is the product of a special orthogonal or unitary group with restrictions of general linear groups.

More explicitly, if π is not a discrete series representation and we write

$$\tilde{\phi}_\pi^{\mathrm{GL}} = \bigoplus_{i \in I_{\tilde{\phi}}^+} m_i \phi_i \oplus \bigoplus_{i \in I_{\tilde{\phi}}^-} 2m_i \phi_i \oplus \bigoplus_{i \in J_{\tilde{\phi}}} m_i(\phi_i \oplus \phi_i^\theta),$$

as in §2.2, then either $m_i > 1$ for some $i \in I_{\tilde{\phi}}^+$ or $m_i > 0$ for some $i \in I_{\tilde{\phi}}^- \cup J_{\tilde{\phi}}$. In either case we can write

$$\tilde{\phi}_\pi^{\mathrm{GL}} = \phi_\tau + \tilde{\phi}_{\pi_0}^{\mathrm{GL}} + \phi_\tau^\theta,$$

where $\tau \in \Pi_{2,\mathrm{temp}}(\mathrm{GL}_{d,K_1})$ is a discrete series representation and $\pi_0 \in \Pi_{\mathrm{temp}}(G(n(G) - [K_1 : K]d))$ is tempered. Then it follows from the local intertwining relations, Theorem 2.3.1, that π is a subrepresentation of $\mathrm{I}_{P_L}^G(\tau \otimes \pi_0)$, where $P_L \leq G$ is a maximal parabolic subgroup with Levi factor

$$M_L \cong \mathrm{Res}_{K_1/K}\mathrm{GL}_d \times G(n(G) - [K_1 : K]d).$$

If M_L is proper, its classical-group factor has geometric rank strictly smaller than $n(G)$. The induction hypothesis applied to σ, followed by transitivity of the canonical L-homomorphisms, proves the assertion for π. Thus it remains only to treat the case in which π is a discrete series representation.

Write

$$M \cong \mathrm{Res}_{K_1/K}\mathrm{GL}_{d_1} \times \cdots \times \mathrm{Res}_{K_1/K}\mathrm{GL}_{d_r} \times G(n_0)$$

and

$$\pi_M = \tau_1 \otimes \cdots \otimes \tau_r \otimes \pi_0 \in \Pi_{\mathrm{sc}}(M).$$

Let

$$\begin{aligned} M_\diamond^{\mathrm{GL}} := \mathrm{Res}_{K_1/K}\mathrm{GL}_{d_1} \times \cdots \times \mathrm{Res}_{K_1/K}\mathrm{GL}_{d_r} \times \mathrm{Res}_{K_1/K}\mathrm{GL}_{\frac{N(G)}{n(G)}n_0} \\ \times \mathrm{Res}_{K_1/K}\mathrm{GL}_{d_r} \times \cdots \times \mathrm{Res}_{K_1/K}\mathrm{GL}_{d_1}, \end{aligned}$$

which is a Levi subgroup of G^{GL}. The diagram (2.3), together with (2.4), identifies the image of $\tilde{\phi}_{\pi_M}^{\mathrm{ss}}$ in $\tilde{\Phi}^{\mathrm{ss}}(G)$ with a semisimple parameter for $M_\diamond^{\mathrm{GL}}$.

Let

$$\tau_{M_\diamond} \in \Pi(M_\diamond^{\mathrm{GL}}) \quad \text{and} \quad \tau_G \in \Pi(G^{\mathrm{GL}})$$

be the representations corresponding, respectively, to

$$(\tilde{\phi}_{\pi_M})_\diamond^{\mathrm{ss,GL}} \quad \text{and} \quad \tilde{\phi}_\pi^{\mathrm{ss,GL}}.$$

These representations need not be tempered. By Lemma 2.4.2, the full general linear transfers associated with π_M and π have the same cuspidal support. In the general linear case, replacing a Weil–Deligne parameter by its semisimplification does not change the cuspidal support. Hence $\tau_{M_\diamond}$ and τ_G have the same cuspidal support. Compatibility of semisimplified general linear parameters with parabolic induction therefore gives

$$(\tilde{\phi}_{\pi_M})_\diamond^{\mathrm{ss,GL}} = \tilde{\phi}_\pi^{\mathrm{ss,GL}}.$$

Finally, the standard representation determines the corresponding class in $\tilde{\Phi}^{\mathrm{ss}}(G)$, including in Case O2 after passage to $\mathrm{O}_{N(G)}(\mathbb{C})$-conjugacy. Thus

$$\tilde{\phi}_\pi^{\mathrm{ss}} = \iota_{M,G}^L \circ \tilde{\phi}_{\pi_M}^{\mathrm{ss}},$$

as required. □

2.5. **Supercuspidal L-packets.** We recall a result of Mœglin and Tadić that characterizes supercuspidal L-parameters in terms of their corresponding L-packets.

Let $\tilde{\phi} \in \tilde{\Phi}_2(G^*)$ be a discrete L-parameter. We write $\mathrm{Jord}(\tilde{\phi})$ for the set of irreducible subrepresentations of $W_{K_1} \times \mathrm{SL}_2(\mathbb{C})$ contained in $\tilde{\phi}^{\mathrm{GL}}$.

An L-parameter $\tilde{\phi}$ is said to be *without gaps* if for every $\rho \boxtimes \mathrm{sp}_a \in \mathrm{Jord}(\tilde{\phi})$ with $a > 2$, one also has $\rho \boxtimes \mathrm{sp}_{a-2} \in \mathrm{Jord}(\tilde{\phi})$.

We recall the following characterization of supercuspidal representations of $G(K)$:

Proposition 2.5.1. *Suppose $\tilde{\phi} \in \tilde{\Phi}_2(G^*)$ is a discrete L-parameter and $\eta \in \mathrm{Irr}(\overline{\mathfrak{S}}_{\tilde{\phi}})$ view as a character of $\mathfrak{S}_{\tilde{\phi}}$ via inflation., then $\tilde{\pi}_{\mathfrak{w},\mathbf{1}}(\tilde{\phi}, \eta) \in \tilde{\Pi}_{\tilde{\phi}}(G^*)$ is supercuspidal if and only if the following two conditions hold:*

- *$\tilde{\phi}$ is without gaps,*
- *$\eta(e_{\rho\boxtimes\mathrm{sp}_a}) = -\eta(e_{\rho\boxtimes\mathrm{sp}_{a-2}})$ for each $\rho \boxtimes \mathrm{sp}_a \in \mathrm{Jord}(\tilde{\phi})$ with $a \geq 2$, where we assume that $\eta(e_{\rho\boxtimes\mathrm{sp}_0}) = 1$.*

Proof. This is established in Case O in [Moe02, MT02], see also [Moe11, Theorem 1.5.1] and [Xu17, Theorem 3.3], and in Case U in [Moe07, Theorem 8.4.4]. Note that by endoscopic character identities Theorem 2.3.4, the LLC defined in [Mok15] is the same as the LLC defined in [Moe07]; cf. [MHN24, Proposition 2.10]. □

We then obtain the following corollary characterizing supercuspidal L-packets:

Corollary 2.5.2. *For a discrete L-parameter $\tilde{\phi} \in \tilde{\Phi}_2(G^*)$, $\tilde{\Pi}_{\tilde{\phi}}(G^*)$ consists of supercuspidal representations if and only if $\tilde{\phi}$ is a supercuspidal L-parameter.*

Proof. If $\tilde{\phi}$ is supercuspidal, then clearly it is without gaps, and it follows from Proposition 2.5.1 that $\tilde{\Pi}_{\tilde{\phi}}(G^*)$ consists of supercuspidal representations.

Conversely, suppose $\tilde{\Pi}_{\tilde{\phi}}(G^*)$ consists of supercuspidal representations, then $\tilde{\phi}$ is without gaps by Proposition 2.5.1. If $\rho\boxtimes\mathrm{sp}_a \in \mathrm{Jord}(\tilde{\phi})$ and $a \geq 2$, then there exists a character η of $\mathfrak{S}_{\tilde{\phi}}$ such that $\eta(z_{\tilde{\phi}}) = 1$ and $\eta(e_{\rho\boxtimes\mathrm{sp}_a}) = \eta(e_{\rho\boxtimes\mathrm{sp}_{a-2}})$. So it follows from Proposition 2.5.1 that $\tilde{\pi}_{\mathfrak{w},\mathbf{1}}(\tilde{\phi}, \eta) \in \tilde{\Pi}_{\tilde{\phi}}(G^*)$ is not supercuspidal, contradiction. Thus we conclude that $a = 1$ for each $\rho \boxtimes \mathrm{sp}_a \in \mathrm{Jord}(\tilde{\phi})$, which means that $\tilde{\phi}$ is supercuspidal. □

2.6. **Combinatorics on L-parameters.** Following [MHN24, §2.2.4], we give a combinatorial description of discrete L-packets that will be used in the explicit computations in the proof of the Kottwitz conjecture.

For $b \in B(G)_{\mathrm{bas}}$, if $\tilde{\phi} \in \tilde{\Phi}_2(G^*)$ is a discrete L-parameter and $\tilde{\pi} \in \tilde{\Pi}_{\tilde{\phi}}(G), \tilde{\rho} \in \tilde{\Pi}_{\tilde{\phi}}(G_b)$, we can define a character

$$\delta[\tilde{\pi}, \tilde{\rho}] = \iota_{\mathfrak{w},b_0}(\tilde{\pi})^\vee \otimes \iota_{\mathfrak{w},b_0+b}(\tilde{\rho}) \in \mathrm{Irr}(\mathfrak{S}_{\tilde{\phi}}),$$

which can be thought of as measuring the relative position of $\tilde{\pi}$ and $\tilde{\rho}$. This character is independent of the Whittaker datum $\mathfrak{w}$ and also b_0, because changing those shifts $\iota_{\mathfrak{w},b_0}$ by a character of $\mathfrak{S}_{\tilde{\phi}}$ [HKW22, Lemma 2.3.3].

Let $b_1 \in B(G)_{\mathrm{bas}}$ be the unique nontrivial basic element. Because $\tilde{\phi}$ is discrete, the packet $\tilde{\Pi}_{\tilde{\phi}}(G)$ (resp. $\tilde{\Pi}_{\tilde{\phi}}(G_{b_1})$) has size $\#\mathfrak{S}_{\tilde{\phi}}/2$ and corresponds via $\iota_{\mathfrak{w},b_0}$ (resp. $\iota_{\mathfrak{w},b_0+b_1}$) to those characters η of $\mathfrak{S}_{\tilde{\phi}}$ such that $\eta(z_{\tilde{\phi}}) = \kappa_{b_0}(-\mathbf{1})$ (resp. $\eta(z_{\tilde{\phi}}) = -\kappa_{b_0}(-\mathbf{1})$).

Write

$$\tilde{\phi}^{\mathrm{GL}} = \phi_1 \oplus \cdots \oplus \phi_k \oplus \phi_{k+1} \oplus \cdots \oplus \phi_r,$$

where the ϕ_i are irreducible representations of $W_{K_1} \times \mathrm{SL}_2(\mathbb{C})$ and $\dim \phi_i$ is odd if and only if $i \leq k$. For subsets $I, J \subset [r]_+$, define their symmetric difference by

$$I \oplus J = (I \smallsetminus J) \cup (J \smallsetminus I),$$

In Case O2, define an equivalence relation $\sim_k$ on $\mathscr{P}([r]_+)$ by

$$I_1 \sim_k I_2 \iff I_1 = I_2 \quad \text{or} \quad I_1 = I_2 \oplus [k]_+.$$

Then we can define the symmetric differences of equivalence classes in $\mathscr{P}([r]_+)/\sim_k$. Note that we can talk about parity of cardinality of equivalence classes of $\mathscr{P}([r]_+)$, because k is always an even number in Case O2.

To unify notation, in Case O1 or Case U, let $\sim_k$ be the trivial equivalence relation. There is a bijection

$$\mathscr{P}([r]_+)/\sim_k \xrightarrow{\sim} \mathrm{Irr}(\mathfrak{S}_{\tilde{\phi}}), \qquad [I] \longmapsto \eta_{[I]} := \sum_{i\in I} e_i^\vee.$$

via the map

$$[I] \in \mathscr{P}([r]_+)/\sim_k \mapsto \eta_{[I]} := \sum_{i\in I} e_i^\vee \in \mathrm{Irr}(\mathfrak{S}_{\tilde{\phi}}).$$

The right-hand side is independent of the representative I of $[I]$. If

$$\#[I] \equiv \frac{\kappa_{b_0}(-\mathbf{1})-1}{2} (\mathrm{mod}\ 2),$$

set

$$\tilde{\pi}_{[I]} := \tilde{\pi}_{\mathfrak{w},b_0}(\tilde{\phi}, \eta_{[I]}).$$

If

$$\#[I] \equiv \frac{\kappa_{b_0}(-\mathbf{1})+1}{2} (\mathrm{mod}\ 2),$$

set

$$\tilde{\pi}_{[I]} := \tilde{\pi}_{\mathfrak{w},b_0+b_1}(\tilde{\phi}, \eta_{[I]}).$$

For any $[I],[J] \in \mathscr{P}([r]_+)/\sim_k$, we then have

$$\delta[\tilde{\pi}_{[I]}, \tilde{\pi}_{[J]}] = \eta_{[I\oplus J]} = \sum_{i\in I\oplus J} e_i^\vee.$$

Let $\mu_1 \in X_\bullet(G)$ be the dominant cocharacter given by

$$z \mapsto \begin{cases} \mathrm{diag}(z, \underbrace{1,\dots,1}_{d(G)-1\text{-many}}) & \text{in Case U} \\ \mathrm{diag}(z, \underbrace{1,\dots,1}_{d(G)-2\text{-many}}, z^{-1}) & \text{in Case O} \end{cases}, \tag{2.5}$$

(i.e., $\mu_1 = \omega_1^\vee$ in the standard notation), where in Case U we use the standard realization of $G_{\overline{K}} \cong \mathrm{GL}_{n(G),\overline{K}}$, and in Case O we use the standard realization of $G_{\overline{K}}$ as the subgroup of $\mathrm{SL}_{d(G),\overline{K}}$ preserving the nondegenerate bilinear form on $\overline{K}\{v_1,\dots,v_{d(G)}\}$ defined by $\langle v_i, v_j\rangle = (1+\delta_{i,j})\delta_{i,d(G)+1-j}$.

Then the highest weight tilting module $\mathcal{T}_{\mu_1}$ of $\widehat{G}$ is the representation $\widehat{\mathrm{Std}}_G$ of $\widehat{G}$.

Theorem 2.6.1. *Let $\tilde{\phi} \in \tilde{\Phi}_2(G)$ be a discrete L-parameter with a decomposition*

$$\tilde{\phi}^{\mathrm{GL}} = \phi_1 \oplus \cdots \oplus \phi_k \oplus \phi_{k+1} \oplus \cdots \oplus \phi_r,$$

where the ϕ_i are distinct irreducible representations of $W_{K_1} \times \mathrm{SL}_2(\mathbb{C})$ of dimension d_i, and d_i is odd if and only if $i \le k$. Let $b_1 \in B(G)_{\mathrm{bas}}$ be the unique nontrivial basic element. Suppose that

$$\tilde{\rho} = \tilde{\pi}_{[I]} \in \tilde{\Pi}_{\tilde{\phi}}(G_{b_1}),$$

and write $\tilde{\pi} = \tilde{\pi}_{[J]} \in \tilde{\Pi}_{\tilde{\phi}}(G)$. Then

$$\mathrm{Hom}_{\mathfrak{S}_{\tilde{\phi}}}\left(\delta[\tilde{\pi},\tilde{\rho}], \tilde{\phi}^{\mathrm{GL}}\right) \cong \bigoplus_{\substack{t\in[r]_+ \\ [J]=[I\oplus\{t\}]}} \phi_t$$

as representations of $W_{K_1} \times \mathrm{SL}_2(\mathbb{C})$*.*

In particular, the direct sum on the right has at most one summand, except in Case O2 with $k = 2$*. In that exceptional case,*

$$[I \oplus \{1\}] = [I \oplus \{2\}],$$

and the corresponding summand is $\phi_1 \oplus \phi_2$*.*

Proof. By the preceding description,

$$\delta[\tilde{\pi}, \tilde{\rho}] = \eta_{[I \oplus J]}.$$

The group $\mathfrak{S}_{\tilde{\phi}}$ acts on the summand ϕ_t of $\tilde{\phi}^{\mathrm{GL}}$ through the character $e_t^\vee = \eta_{[\{t\}]}$. Hence

$$\begin{aligned} \mathrm{Hom}_{\mathfrak{S}_{\tilde{\phi}}}(\delta[\tilde{\pi}, \tilde{\rho}], \phi_t) \neq 0 &\iff \eta_{[I \oplus J]} = \eta_{[\{t\}]} \\ &\iff [I \oplus J] = [\{t\}] \\ &\iff [J] = [I \oplus \{t\}]. \end{aligned}$$

This proves the displayed direct-sum formula.

It remains to determine when two different indices can contribute. If

$$[I \oplus \{t\}] = [I \oplus \{u\}] \qquad (t \neq u),$$

then the equivalence relation must be nontrivial, so we are in Case O2, and

$$\{t, u\} = [k]_+.$$

Since the left-hand side has cardinality two, this is possible exactly when $k = 2$ and $\{t, u\} = \{1, 2\}$. Thus the only repeated class contributes $\phi_1 \oplus \phi_2$. $\square$

3. Local Langlands correspondence via moduli of local shtukas

In this section, we review the construction of the Fargues–Scholze local Langlands correspondence.

Let p be a rational prime, let $K/\mathbb{Q}_p$ be a finite extension, and let G be a connected reductive group over K. Let ℓ be a rational prime different from p, and let $\Lambda \in \{\overline{\mathbb{Q}_\ell}, \overline{\mathbb{F}_\ell}\}$ be such that $\#\pi_0(Z(\mathsf{G}))$ is invertible in Λ.[6] Fix an isomorphism $\iota_\ell : \mathbb{C} \to \overline{\mathbb{Q}_\ell}$ sending the chosen complex square root of p to a chosen square root in $\overline{\mathbb{Q}_\ell}$, whose reduction determines a square root in $\overline{\mathbb{F}_\ell}$.

We use L-groups and L-parameters with Λ-coefficients. Let $\Phi(\mathsf{G}, \Lambda)$ denote the set of L-parameters

$$W_K \times \mathrm{SL}_2(\Lambda) \longrightarrow {}^L\mathsf{G}(\Lambda),$$

and let $\Pi(\mathsf{G}, \Lambda)$ denote the set of irreducible smooth representations of $\mathsf{G}(K)$ with coefficients in Λ.

3.1. **The correspondence.** We briefly recall the construction of the Fargues–Scholze local Langlands correspondence. The Kottwitz set $B(\mathsf{G})$, as defined in [Kot85], consists of φ_K-equivalence classes of $\mathsf{G}(\breve{K})$, i.e.,

$$\mathsf{b} \sim \mathsf{b}' \iff \mathsf{b}' = \mathsf{g}^{-1}\mathsf{b}\varphi_K(\mathsf{g}) \text{ for some } \mathsf{g} \in \mathsf{G}(\breve{K}).$$

Each element $\mathsf{b} \in B(\mathsf{G})$ is determined by two invariants: The Kottwitz invariant $\kappa_{\mathsf{G}}(\mathsf{b})$ and the $\mathsf{G}(\breve{K})$-conjugacy class of its slope homomorphism (or Newton map)

$$\nu_{\mathsf{b}} : \mathbf{D}_{\breve{K}} \longrightarrow \mathsf{G}_{\breve{K}},$$

where $\mathbf{D}$ is the pro-algebraic diagonalizable group whose character group is $\mathbb{Q}$.

An element $\mathsf{b} \in B(\mathsf{G})$ is called *basic* if ν_{b} is central in G; the set of basic elements in $B(\mathsf{G})$ is denoted $B(\mathsf{G})_{\mathrm{bas}}$. An element $\mathsf{b} \in B(\mathsf{G})$ is called *unramified* if it lies in the image of the map $B(\mathsf{T}) \to B(\mathsf{G})$ for some maximal torus $\mathsf{T} \leq \mathsf{G}$. Denote by $B(\mathsf{G})_{\mathrm{un}} \subset B(\mathsf{G})$ the subset of unramified

[6] This is the coefficient hypothesis used in [FS24] to avoid additional complications in the ℓ-modular setting.

elements. These are precisely those $\mathsf{b} \in B(\mathsf{G})$ for which the twisted centralizer G_b is quasisplit; see [Ham24, Lemma 2.12].

We recall some material from [SW20] and [Far16] regarding the relative Fargues–Fontaine curve. For any $S \in \mathcal{P}\mathrm{erfd}_\kappa$, the associated Fargues–Fontaine curve X_S is defined as in [FF18]. When $S = \mathrm{Spa}(R, R^+)$ is affinoid with pseudo-uniformizer ϖ, the adic space X_S is defined as follows:

$$Y_S = \mathrm{Spa}(W_{\mathcal{O}_K}(R^+)) \smallsetminus \{p[\varpi] = 0\},$$

$$X_S = Y_S/\varphi_K^{\mathbb{Z}},$$

For any affinoid perfectoid space S over κ, the following sets are canonically in bijection:

(1) $\mathrm{Spd}(K)(S)$,

(2) Untilts $S^\sharp$ of S over K,

(3) Cartier divisors of Y_S of degree 1.

For any untilt $S^\sharp$ of S, we denote by $D_{S^\sharp} \subset Y_S$ the corresponding divisor.

By [FS24, Theorem III.0.2], the presheaf Bun_G on $\mathcal{P}\mathrm{erfd}_{\overline{\kappa}}$, which assigns to each S the groupoid of G-torsors on X_S, is a small Artin v-stack. For any $S \in \mathcal{P}\mathrm{erfd}_{\overline{\kappa}}$, there exists a functor $\mathsf{b} \mapsto \mathcal{E}^\mathsf{b}$ from the category of isocrystals with G-structure to $\mathrm{Bun}_\mathsf{G}(S)$. When $S = \mathrm{Spd}(\overline{\kappa})$, this map induces a bijection from $B(\mathsf{G})$ to the set of isomorphism classes of G-bundles on X_S. More precisely, there exists a homeomorphism $|\mathrm{Bun}_\mathsf{G}| \to B(\mathsf{G})$ sending $\mathcal{E}^\mathsf{b}$ to b; see [Far20, Ans23, Vie24].

For any $\mathsf{b} \in B(\mathsf{G})$, the sub-functor

$$\mathrm{Bun}_\mathsf{G}^\mathsf{b} := \mathrm{Bun}_\mathsf{G} \times_{|\mathrm{Bun}_\mathsf{G}|} \{\mathsf{b}\} \subset \mathrm{Bun}_\mathsf{G}$$

is locally closed. It can be identified with the classifying stack $[\mathrm{Spd}(\overline{\kappa})/\tilde{\mathsf{G}}_\mathsf{b}]$, where $\tilde{\mathsf{G}}_\mathsf{b}$ denotes the v-sheaf of groups given by $S \mapsto \mathrm{Aut}_{X_S}(\mathcal{E}^\mathsf{b})$; see [FS24, Proposition III.5.3]. The natural morphism $\mathrm{Bun}_\mathsf{G}^\mathsf{b} \to [\mathrm{Spd}(\overline{\kappa})/\underline{\mathsf{G}_\mathsf{b}(K)}]$ induces equivalences of categories

$$\mathrm{D}(\mathsf{G}_\mathsf{b}, \Lambda) \cong \mathrm{D}_{\mathrm{lis}}([\mathrm{Spd}(\overline{\kappa})/\underline{\mathsf{G}_\mathsf{b}(K)}], \Lambda) \cong \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_\mathsf{G}^\mathsf{b}, \Lambda),$$

see [FS24, Theorem VII.7.1].

Writing i_b for the inclusion $\mathrm{Bun}_\mathsf{G}^\mathsf{b} \subset \mathrm{Bun}_\mathsf{G}$, any $\pi \in \Pi(\mathsf{G}_\mathsf{b}, \Lambda)$ may be regarded as an object in $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_\mathsf{G}, \Lambda)$ via the extension by zero $(i_\mathsf{b})_!$, which is well-defined by [FS24, Proposition VII.7.3, Proposition VII.6.7]. Moreover, when b is basic, the map $\mathrm{Bun}_\mathsf{G}^\mathsf{b} \to [\mathrm{Spd}(\overline{\kappa})/\underline{\mathsf{G}_\mathsf{b}(K)}]$ is an isomorphism; see [FS24, Proposition III.4.5].

We now introduce the Hecke operators. For each finite index set I, let $\mathrm{Rep}_\Lambda({}^L\mathsf{G}^I)$ denote the category of algebraic representations of I copies of ${}^L\mathsf{G}(\Lambda)$ over Λ, and let Div^I be the I-fold product of the mirror curve $\mathrm{Div}^1 := \mathrm{Spd}(\breve{K})/\varphi_K^{\mathbb{Z}}$. The diamond Div^1 represents the functor that sends $S \in \mathcal{P}\mathrm{erfd}_{\overline{\kappa}}$ to the set of degree 1 Cartier divisors on X_S.

We then have the Hecke correspondence diagram

$$\begin{tikzcd} & \mathrm{Hk}_{\mathsf{G},I} \arrow[dl, "h^{\leftarrow}"'] \arrow[dr, "h^{\rightarrow} \times \mathrm{Supp}"] & \\ \mathrm{Bun}_\mathsf{G} & & \mathrm{Bun}_\mathsf{G} \times \mathrm{Div}^I \end{tikzcd}, \tag{3.1}$$

where $\mathrm{Hk}_{\mathsf{G},I}$ represents the functor sending $S \in \mathcal{P}\mathrm{erfd}_{\overline{\kappa}}$ to the groupoid of tuples

$$(\mathcal{E}_1, \mathcal{E}_2, \beta, (D_i)_{i \in I}),$$

where $D_i \subset X_S$ are Cartier divisors, $\mathcal{E}_1, \mathcal{E}_2$ are G-torsors on X_S, together with an isomorphism

$$\beta : \mathcal{E}_1|_{X_S \setminus \cup_{i\in I} D_i} \xrightarrow{\sim} \mathcal{E}_2|_{X_S \setminus \cup_{i \in I} D_i}.$$

The morphism $h^{\leftarrow}$ sends the tuple to $\mathcal{E}_1$, while $h^{\rightarrow} \times \mathrm{Supp}$ sends it to $(\mathcal{E}_2, (D_i)_{i \in I})$.

For each $W \in \mathrm{Rep}_\Lambda({}^L\mathsf{G}^I)$, Fargues and Scholze define a solid Λ-sheaf $\mathcal{S}'_W \in \mathrm{D}_\blacksquare(\mathrm{Hk}_{\mathsf{G},I}, \Lambda)$ via the geometric Satake correspondence; see [FS24, Theorem I.6.3]. The corresponding Hecke operator

$$\mathrm{T}_W : \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_\mathsf{G}, \Lambda) \to \mathrm{D}_\blacksquare(\mathrm{Bun}_\mathsf{G} \times \mathrm{Div}^I, \Lambda) : A \mapsto \mathrm{R}(h^{\rightarrow} \times \mathrm{Supp})_\natural(h^{\leftarrow *}(A) \otimes^{\mathrm{L}} \mathcal{S}'_W). \tag{3.2}$$

Here $\mathrm{R}(h^{\rightarrow} \times \mathrm{Supp})_\natural$ is the natural push-forward, left adjoint to restriction; [FS24, Proposition VII.3.1]. The results of [FS24, Theorem I.7.2, Proposition IX.2.1, Corollary IX.2.3] show that, after taking the geometric fiber over Div^I used in the construction of excursion operators, this object is lisse and carries a canonical continuous W_K^I-action. By abuse of notation, we therefore also write

$$\mathrm{T}_W : \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_\mathsf{G}, \Lambda) \longrightarrow \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_\mathsf{G}, \Lambda)^{\mathrm{B}W_K^I}$$

for the resulting functor.

For any $\pi \in \Pi(\mathsf{G}, \Lambda)$, the Fargues–Scholze L-parameter ϕ_π^{FS} is extracted from the action of excursion operators on π. Consider a tuple

$$(I, W, (\gamma_i)_{i \in I}, \alpha, \beta),$$

where

- I is a finite index set, and $\Delta_I : {}^L\mathsf{G} \to {}^L\mathsf{G}^I$ for the diagonal embedding;
- $(r_W, W) \in \mathrm{Rep}_\Lambda({}^L\mathsf{G}^I)$ is an algebraic representation;
- $\gamma_i \in W_K$ for every $i \in I$;
- $$\alpha : \mathbf{1} \longrightarrow \Delta_I^* W, \qquad \beta : \Delta_I^* W \longrightarrow \mathbf{1}$$
  are morphisms in $\mathrm{Rep}_\Lambda({}^L\mathsf{G})$.

The associated excursion operator is the natural transformation of the identity functor given by

$$\mathrm{id} = \mathrm{T}_\mathbf{1} \xrightarrow{\mathrm{T}_\alpha} \mathrm{T}_{\Delta_I^* W} \cong \mathrm{T}_W \xrightarrow{(\gamma_i)_{i \in I}} \mathrm{T}_W \cong \mathrm{T}_{\Delta_I^* W} \xrightarrow{\mathrm{T}_\beta} \mathrm{T}_\mathbf{1} = \mathrm{id}.$$

Evaluating this natural transformation at π gives a scalar by Schur's lemma. The relations among these scalars, together with Lafforgue's reconstruction theorem [Laf18, Proposition 11.7], determine the Fargues–Scholze parameter $\phi_\pi^{\mathrm{FS}} : W_K \to {}^L\mathsf{G}(\Lambda)$ characterized by the requirement that the above excursion operator acts on π through the scalar defined by

$$\Lambda \xrightarrow{\alpha} W \xrightarrow{r_W\left(\left(\phi_\pi^{\mathrm{FS}}(\gamma_i)\right)_{i \in I}\right)} W \xrightarrow{\beta} \Lambda.$$

See [FS24, Definition/Proposition IX.4.1].

Then Fargues and Scholze [FS24, Theorem I.9.6] showed that their construction has various desirable properties:

Theorem 3.1.1. *The Fargues–Scholze LLC* $\mathrm{rec}_\mathsf{G}^{\mathrm{FS}} : \Pi(\mathsf{G}, \Lambda) \to \Phi^{\mathrm{ss}}(\mathsf{G}, \Lambda)$ *satisfies the following compatibility properties:*

1. *(Compatibility with local class field theory) If* G *is a torus, then* $\mathrm{rec}_\mathsf{G}^{\mathrm{FS}}$ *is the usual local Langlands correspondence for tori constructed from local class field theory,*
2. *(Compatibility with natural operations)* $\mathrm{rec}_\mathsf{G}^{\mathrm{FS}}$ *is compatible with character twists, central characters, and taking contragredients,*
3. *(Compatibility with products) If* $\mathsf{G} = \mathsf{G}_1 \times \mathsf{G}_2$ *is a product of two groups and* $\pi_i \in \Pi(\mathsf{G}_i, \Lambda)$ *for each* $i \in \{1, 2\}$*, then* $\mathrm{rec}_\mathsf{G}^{\mathrm{FS}}(\pi_1 \boxtimes \pi_2) = \mathrm{rec}_{\mathsf{G}_1}^{\mathrm{FS}}(\pi_1) \times \mathrm{rec}_{\mathsf{G}_2}^{\mathrm{FS}}(\pi_2)$*,*
4. *(Compatibility with central extensions) If* $\mathsf{G}' \to \mathsf{G}$ *is a map of reductive groups inducing an isomorphism on adjoint groups,* $\pi \in \Pi(\mathsf{G}, \Lambda)$*, and* $\pi' \in \Pi(\mathsf{G}', \Lambda)$ *is an irreducible admissible subquotient of* $\pi|_{\mathsf{G}'(K)}$*, then* $\mathrm{rec}_{\mathsf{G}'}^{\mathrm{FS}}(\pi')$ *is given by* $\mathrm{rec}_\mathsf{G}^{\mathrm{FS}}(\pi)$ *composed with the natural map* ${}^L G(\Lambda) \to {}^L G'(\Lambda)$*. In particular,* $\mathrm{rec}_{\mathsf{G}'}^{\mathrm{FS}}$ *commutes with contragredients and Chevalley involutions.*

(5) *(Compatibility with parabolic induction) If* $\mathsf{P} \le \mathsf{G}$ *is a parabolic subgroup with Levi factor* M *and* $\pi_{\mathsf{M}} \in \Pi(\mathsf{M}, \Lambda)$*, then for any subquotient* π *of the normalized parabolic induction* $\mathrm{I}_{\mathsf{P}}^{\mathsf{G}}(\pi_{\mathsf{M}})$, $\mathrm{rec}_{\mathsf{G}}^{\mathrm{FS}}(\pi)$ *is given by the composition*

$$W_K \xrightarrow{\mathrm{rec}_{\mathsf{M}}^{\mathrm{FS}}(\pi_{\mathsf{M}})} {}^L\mathsf{M}(\Lambda) \to {}^L\mathsf{G}(\Lambda),$$

where ${}^L\mathsf{M}(\Lambda) \to {}^L\mathsf{G}(\Lambda)$ *is the canonical embedding.*

(6) *(Compatibility with Harris–Taylor/Henniart LLC) If* $\Lambda = \overline{\mathbb{Q}_\ell}$ *and* $\mathsf{G} = \mathrm{GL}_n$*, then* $\mathrm{rec}_{\mathrm{GL}_n}^{\mathrm{FS}}$ *coincides with the (semisimplified) local Langlands correspondence given by Harris–Taylor and Henniart in the sense of* Theorem A.

(7) *(Compatibility with Weil restriction) If* $\mathsf{G} = \mathrm{Res}_{K'/K}\mathsf{G}'$ *for some reductive group* G' *over a finite extension* K'/K*, then* $\mathrm{rec}_{\mathsf{G}'}^{\mathrm{FS}}(\pi) = \mathrm{rec}_{\mathsf{G}}^{\mathrm{FS}}(\pi)|_{W_{K'}}$ *for any* $\pi \in \Pi(\mathsf{G}', \Lambda) = \Pi(\mathsf{G}, \Lambda)$.

(8) *(Compatibility with contragredients) For* $\pi \in \Pi(\mathsf{G}, \Lambda)$*, we have* $\mathrm{rec}_{\mathsf{G}}^{\mathrm{FS}}(\pi^\vee) = \vartheta_{\widehat{\mathsf{G}}}\left(\mathrm{rec}_{\mathsf{G}}^{\mathrm{FS}}(\pi)\right)$*; see* [FS24, Proposition IX.5.3]. *Here we recall that for each Borel pair* $(\widehat{\mathsf{B}}, \widehat{\mathsf{T}})$ *of* $\widehat{\mathsf{G}}$*, there exists a Cartan involution (also called the Chevalley involution)* $\vartheta_{\widehat{\mathsf{G}}}$ *of* $\widehat{\mathsf{G}}$ *which preserves the fixed pinning and acts as* $t \mapsto w_0(t^{-1})$ *on* $\widehat{\mathsf{T}}$*, where* w_0 *is the longest Weyl element taking* $\widehat{\mathsf{B}}$ *to the opposite Borel group.* $\vartheta_{\widehat{\mathsf{G}}}$ *extends to an involution of* ${}^L\mathsf{G}$ *because the action of* W_K *on* $\widehat{\mathsf{G}}$ *preserves the pinning.*

(9) *(Compatibility with reduction modulo* ℓ*) If* $\pi \in \Pi(\mathsf{G}, \overline{\mathbb{Q}_\ell})$ *admits a* $\mathsf{G}(K)$*-stable* $\overline{\mathbb{Z}_\ell}$*-lattice, and the reduction modulo* ℓ *representation of* π *has an irreducible subquotient* $\overline{\pi}'$*, then* $\mathrm{rec}_{\mathsf{G}}^{\mathrm{FS}}(\pi) : W_K \to {}^L\mathsf{G}(\overline{\mathbb{Q}_\ell})$ *factors through* ${}^L\mathsf{G}(\overline{\mathbb{Z}_\ell})$*, and its reduction modulo* ℓ *equals* $\mathrm{rec}_{\mathsf{G}}^{\mathrm{FS}}(\overline{\pi}')$*; see* [FS24, §IX.5.2].

Moreover, Hansen, Kaletha and Weinstein established in [HKW22, Theorem 6.6.1] that the Fargues–Scholze LLC coincides with the usual (semisimplified) LLC for inner forms of general linear groups given in [DKV84, Rog83] via the Jacquet–Langlands correspondence.

When $\mathsf{G}^\sharp$ is an inner form of GL_n and $\mathsf{G} = [\mathsf{G}^\sharp, \mathsf{G}^\sharp]$, the local Langlands correspondence constructed in [GK82, Tad92, HS12, ABPS16] assigns to each $\pi^\sharp \in \Pi(\mathsf{G}^\sharp)$ a parameter $\phi_{\pi^\sharp}$ with the property that, for any irreducible admissible subrepresentation $\pi \subset \pi^\sharp|_{\mathsf{G}(K)}$, ϕ_π equals $\phi_{\pi^\sharp}$ composed with the natural map ${}^L\mathsf{G}^\sharp \to {}^L\mathsf{G}$. This prescription uniquely determines the local Langlands correspondence for G, since every $\pi \in \Pi(\mathsf{G})$ occurs as a subrepresentation of $\pi^\sharp|_{\mathsf{G}(K)}$ for some $\pi^\sharp \in \Pi(\mathsf{G}^\sharp)$, by [GK82, Lemma 2.3].

Combining the compatibility of the Fargues–Scholze LLC with the LLC for inner forms of general linear groups with its compatibility under homomorphisms inducing an isomorphism on adjoint groups (see Theorem 3.1.1), we obtain the following.

Proposition 3.1.2. *Let* G *be an inner form of* SL_N *over* K *for some positive integer* $N \in \mathbb{Z}_+$*, and suppose that* $\Lambda = \overline{\mathbb{Q}_\ell}$*. Then the Fargues–Scholze LLC is compatible with the LLC constructed in* [GK82, Tad92, HS12, ABPS16] *in the sense of* Theorem A.

Proof. Choose an inner form $\mathsf{G}^\sharp$ of GL_N such that $[\mathsf{G}^\sharp, \mathsf{G}^\sharp] = \mathsf{G}$, and let

$$q : {}^L\mathsf{G}^\sharp \longrightarrow {}^L\mathsf{G}$$

be the natural map. Let $\pi \in \Pi(\mathsf{G})$. By [GK82, Lemma 2.3], there exists $\pi^\sharp \in \Pi(\mathsf{G}^\sharp)$ such that π is an irreducible constituent of $\pi^\sharp|_{\mathsf{G}(K)}$. By the construction of the classical LLC for inner forms of SL_N, $\mathrm{rec}_{\mathsf{G}}(\pi) = q \circ \mathrm{rec}_{\mathsf{G}^\sharp}(\pi^\sharp)$ up to $\widehat{\mathsf{G}}$-conjugacy.

On the other hand, the compatibility of the Fargues–Scholze LLC under maps inducing an isomorphism on adjoint groups, as stated in Theorem 3.1.1, gives

$$\mathrm{rec}_{\mathsf{G}}^{\mathrm{FS}}(\pi) = q \circ \mathrm{rec}_{\mathsf{G}^\sharp}^{\mathrm{FS}}(\pi^\sharp).$$

By [HKW22, Theorem 6.6.1], the Fargues–Scholze LLC for $\mathsf{G}^\sharp$ agrees with the semisimplification of the classical LLC. Consequently,

$$\begin{aligned}\mathrm{rec}^{\mathrm{FS}}_{\mathsf{G}}(\pi) &= q \circ \mathrm{rec}^{\mathrm{FS}}_{\mathsf{G}^\sharp}(\pi^\sharp)\\ &= q \circ \mathrm{rec}_{\mathsf{G}^\sharp}(\pi^\sharp)^{\mathrm{ss}}\\ &= \left(q \circ \mathrm{rec}_{\mathsf{G}^\sharp}(\pi^\sharp)\right)^{\mathrm{ss}}\\ &= \mathrm{rec}_{\mathsf{G}}(\pi)^{\mathrm{ss}}.\end{aligned}$$

This is precisely the compatibility asserted in Theorem A. $\square$

3.2. **Local shtuka spaces.** In this subsection we introduce the local shtuka spaces whose cohomology will be one of the main objects of study of this paper.

Let G^* be the quasisplit inner form of G with a fixed Borel subgroup B^* containing a maximal torus T^*, and let μ be a dominant cocharacter of $\mathsf{G}^*_{\overline{K}}$ with reflex field E_μ. We first recall the notion of neutral μ-acceptable elements from [RV14, Definition 2.3]: Let $\mu^\natural$ be the image of μ in $X^\bullet\left(Z(\widehat{\mathsf{G}^*})^{\mathrm{Gal}_K}\right)$, and let

$$\overline{\mu} := \frac{1}{[E_\mu : K]} \sum_{\gamma \in \mathrm{Gal}_K / \mathrm{Gal}_{E_\mu}} \gamma(\mu) \in X_\bullet(\mathsf{T}^*)^{+,\mathrm{Gal}_K}_{\mathbb{Q}}.$$

For $\mathsf{b}_0 \in B(\mathsf{G}^*)$, define the set of neutral μ-acceptable elements by

$$B(\mathsf{G}^*, \mathsf{b}_0, \mu) := \{\mathsf{b} \in B(\mathsf{G}^*) | \kappa_{\mathsf{G}^*}(\mathsf{b}) - \kappa_{\mathsf{G}^*}(\mathsf{b}_0) = \mu^\natural, \quad \overline{\mu} - (\nu_{\mathsf{b}} - \nu_{\mathsf{b}_0}) \in \mathbb{R}_+\Phi^\vee(\mathsf{G}^*, \mathsf{T}^*)^+\} \tag{3.3}$$

where we regard the Newton maps $\nu_{\mathsf{b}}, \nu_{\mathsf{b}_0} : \mathbf{D}(\mathbb{Q}) \to \mathsf{G}^*_{\overline{K}}$ as elements of $X_\bullet(\mathsf{T}^*)^{+,\mathrm{Gal}_K}_{\mathbb{Q}}$.

For example, for the quasisplit group G^* defined in §2.1 with a pure inner twist $(G = G^*_{b_0}, \varrho_{b_0}, z_{b_0})$ and the geometric cocharacter $\mu_1 \in X_\bullet(G^*)$ defined in §2.6, the unique basic element $b_1 \in B(G^*, b_0, \mu_1)_{\mathrm{bas}}$ is just the unique nontrivial basic element b_1 of $B(G)_{\mathrm{bas}}$ under the isomorphism (2.2).

We then recall Scholze's definition [SW20, §23] of the local shtuka space in the basic case: For each $\mathsf{b}_0 \in B(\mathsf{G}^*)_{\mathrm{bas}}$ and $\mathsf{b} \in B(\mathsf{G}^*, \mathsf{b}_0, \mu)_{\mathrm{bas}}$, the local shtuka space $\mathrm{Sht}(\mathsf{G}^*, \mathsf{b}, \mathsf{b}_0, \mu)$ is a local spatial diamond over $\mathrm{Spd}(\breve{K}E_\mu)$ that represents the functor that maps $S \in \mathcal{P}\mathrm{erfd}_{\breve{K}E_\mu}$ to the set of isomorphisms

$$\gamma : \mathcal{E}^{\mathsf{b}}|_{X_{S^\flat} \smallsetminus D_S} \xrightarrow{\sim} \mathcal{E}^{\mathsf{b}_0}|_{X_{S^\flat} \smallsetminus D_S},$$

of G^*-bundles that are meromorphic along the divisor $D_S \subset X_{S^\flat}$ and bounded by μ pointwise on $\mathrm{Spa}(S)$, as defined in [HKW22, p. 11].[7]

The automorphism groups of $\mathcal{E}^{\mathsf{b}}$ and $\mathcal{E}^{\mathsf{b}_0}$ are the constant group diamonds $\underline{\mathsf{G}^*_{\mathsf{b}}(K)}$ and $\underline{\mathsf{G}^*_{\mathsf{b}_0}(K)}$, respectively, so $\mathrm{Sht}(\mathsf{G}^*, \mathsf{b}, \mathsf{b}_0, \mu)$ is equipped with a commuting action of $\underline{\mathsf{G}^*_{\mathsf{b}}(K)}$ and $\underline{\mathsf{G}^*_{\mathsf{b}_0}(K)}$ by pre- and post-composing on γ.

We record the compatibility of the local shtuka space under Weil restrictions. Let K/K_0 be an unramified extension and set $\mathsf{G}^*_0 = \mathrm{Res}_{K/K_0}\mathsf{G}^*$. Suppose that, under the identification

$$(\mathsf{G}^*_0)_{\overline{K}_0} \cong \prod_{\mathrm{Hom}(K, \overline{K}_0)} \mathsf{G}^*_{\overline{K}},$$

the cocharacter μ_0 is trivial on every factor except one, where it equals a cocharacter μ of $\mathsf{G}^*_{\overline{K}}$. Then μ_0 and μ have the same reflex field. For $\mathsf{b}_0 \in B(\mathsf{G}^*_0)_{\mathrm{bas}}$ and $\mathsf{b} \in B(\mathsf{G}^*_0, \mathsf{b}_0, \mu_0)_{\mathrm{bas}}$, use the natural

[7]We note that our definition of $\mathrm{Sht}(\mathsf{G}^*, \mathsf{b}, \mathsf{b}_0, \mu)$ agrees with the definition in [Ham25, §3.1], [MHN24, §2.3.2], [Ham24, §10], but differs from the definitions in [SW20] and [HKW22, Definition 2.4.1] by changing μ to μ^{-1}. Our definition simplifies certain presentations.

isomorphism $B(\mathsf{G}_0^*) \cong B(\mathsf{G}^*)$ to regard b_0 and b as basic elements of $B(\mathsf{G}^*)$. The definitions then give a natural isomorphism of diamonds

$$\mathrm{Sht}(\mathsf{G}^*, \mathsf{b}, \mathsf{b}_0, \mu) \cong \mathrm{Sht}(\mathsf{G}_0^*, \mathsf{b}, \mathsf{b}_0, \mu_0).$$

For each compact open subgroup $\mathscr{K} \leq \mathsf{G}^*_{\mathsf{b}_0}(K)$, we define $\mathrm{Sht}_{\mathscr{K}}(\mathsf{G}^*, \mathsf{b}, \mathsf{b}_0, \mu) = \mathrm{Sht}(\mathsf{G}^*, \mathsf{b}, \mathsf{b}_0, \mu)/\mathscr{K}$, which is also a locally spatial diamond defined over $\mathrm{Spd}(\breve{K}E_\mu)$ [SW20, Theorem 23.1.4]. Let $\mathcal{S}'_\mu$ be the Λ-sheaf corresponding to the highest weight Tilting module $\mathcal{T}_\mu \in \mathrm{Rep}_\Lambda(\widehat{\mathsf{G}^*})$ via the geometric Satake equivalence [FS24, Theorem I.6.3]. There is a natural map

$$\mathrm{pr} : \mathrm{Sht}(\mathsf{G}^*, \mathsf{b}, \mathsf{b}_0, \mu) \to \mathrm{Hk}_{\mathsf{G}^*,1},$$

and we denote the pullback of $\mathcal{S}'_\mu$ along pr by $\mathcal{S}_\mu$. Note that pr factors through the quotient of $\mathrm{Sht}(\mathsf{G}^*, \mathsf{b}, \mathsf{b}_0, \mu)$ by the actions of $\mathsf{G}^*_{\mathsf{b}}(K)$ and $\mathsf{G}^*_{\mathsf{b}_0}(K)$, so $\mathcal{S}_\mu$ is equivariant with respect to these actions. We define

$$\mathrm{R}\Gamma_c(\mathrm{Sht}_{\mathscr{K}}(\mathsf{G}^*, \mathsf{b}, \mathsf{b}_0, \mu), \mathcal{S}_\mu) := \varinjlim_U \mathrm{R}\Gamma_c(U, \mathcal{S}_\mu)$$

where U runs through quasi-compact open subsets of $\mathrm{Sht}_{\mathscr{K}}(\mathsf{G}^*, \mathsf{b}, \mathsf{b}_0, \mu)$, and also

$$\mathrm{R}\Gamma_c(\mathsf{G}^*, \mathsf{b}, \mathsf{b}_0, \mu) := \varinjlim_{\mathscr{K}} \mathrm{R}\Gamma_c(\mathrm{Sht}_{\mathscr{K}}(\mathsf{G}^*, \mathsf{b}, \mathsf{b}_0, \mu), \mathcal{S}_\mu)$$

where $\mathscr{K}$ runs through open compact subgroups of $\mathsf{G}^*_{\mathsf{b}_0}(K)$.[8]

Following [Shi11], we now define a map $\mathrm{Mant}_{\mathsf{G}^*,\mathsf{b},\mathsf{b}_0,\mu} : \mathrm{K}_0(\mathsf{G}^*_{\mathsf{b}}, \Lambda) \to \mathrm{K}_0(\mathsf{G}^*_{\mathsf{b}_0}(K) \times W_{E_\mu}, \Lambda)$ describing the cohomology of $\mathrm{Sht}(\mathsf{G}^*, \mathsf{b}, \mathsf{b}_0, \mu)$, where for each $\mathsf{b} \in B(\mathsf{G}^*)_{\mathrm{bas}}$, we denote by $\mathrm{K}_0(\mathsf{G}^*_{\mathsf{b}}, \Lambda)$ the Grothendieck group of the category of finite-length admissible representations of $\mathsf{G}^*_{\mathsf{b}}(K)$ with Λ-coefficients, and denote by $\mathrm{K}_0(\mathsf{G}^*_{\mathsf{b}}(K) \times W_{E_\mu}, \Lambda)$ the Grothendieck group of the category of finite-length admissible representations of $\mathsf{G}^*_{\mathsf{b}}(K)$ with Λ-coefficients equipped with a continuous action of W_{E_μ} commuting with the $\mathsf{G}^*_{\mathsf{b}}(K)$-action.

Definition 3.2.1. For $\rho \in \Pi(\mathsf{G}^*_{\mathsf{b}}, \Lambda)$, define

$$\mathrm{R}\Gamma^\flat_c(\mathsf{G}^*, \mathsf{b}, \mathsf{b}_0, \mu)[\rho] := \mathrm{RHom}_{\mathsf{G}^*_{\mathsf{b}}(K)}\left(\mathrm{R}\Gamma_c(\mathsf{G}^*, \mathsf{b}, \mathsf{b}_0, \mu), \rho\right), \tag{3.4}$$

and

$$\mathrm{R}\Gamma_c(\mathsf{G}^*, \mathsf{b}, \mathsf{b}_0, \mu)[\rho] := \mathrm{R}\Gamma_c(\mathsf{G}^*, \mathsf{b}, \mathsf{b}_0, \mu) \otimes^{\mathrm{L}}_{\mathsf{G}^*_{\mathsf{b}}(K)} \rho.$$

It follows from [FS24, Corollary I.7.3 and p. 317] that $\mathrm{R}\Gamma^\flat_c(\mathsf{G}^*, \mathsf{b}, \mathsf{b}_0, \mu)[\rho]$ is a finite-length W_{E_μ}-equivariant object of $\mathrm{D}(\mathsf{G}^*_{\mathsf{b}_0}, \Lambda)$. We denote its image in $\mathrm{K}_0(\mathsf{G}^*_{\mathsf{b}_0}(K) \times W_{E_\mu}; \Lambda)$ by

$$\mathrm{Mant}_{\mathsf{G}^*,\mathsf{b},\mathsf{b}_0,\mu}(\rho).$$

Note that $\mathrm{R}\Gamma_c(\mathsf{G}^*, \mathsf{b}, \mathsf{b}_0, \mu)[\rho]$ is much more natural from the point of view of geometric arguments on $\mathrm{Bun}_{\mathsf{G}^*}$ as it involves the much simpler extension by zero functor, while the complex $\mathrm{R}\Gamma^\flat_c(\mathsf{G}^*, \mathsf{b}, \mathsf{b}_0, \mu)[\rho]$ is studied in [HKW22]. It follows from Hom-Tensor duality that

$$\mathrm{R}\Gamma^\flat_c(\mathsf{G}^*, \mathsf{b}, \mathsf{b}_0, \mu)[\rho^*] \cong \mathrm{RHom}\left(\mathrm{R}\Gamma_c(\mathsf{G}^*, \mathsf{b}, \mathsf{b}_0, \mu)[\rho], \Lambda\right). \tag{3.5}$$

Moreover, we have the following result of Meli, Hamann and Nguyen [MHN24, Proposition 2.25] for representations with supercuspidal Fargues–Scholze L-parameters:

Proposition 3.2.2. *If $\rho \in \Pi(\mathsf{G}^*_{\mathsf{b}}, \Lambda)$ has supercuspidal Fargues–Scholze L-parameter ϕ^{FS}_ρ, then there exists an isomorphism*

$$\mathrm{R}\Gamma_c(\mathsf{G}^*, \mathsf{b}, \mathsf{b}_0, \mu)[\rho] \cong \mathrm{R}\Gamma^\flat_c(\mathsf{G}^*, \mathsf{b}, \mathsf{b}_0, \mu)[\rho]$$

of representations of $\mathsf{G}^*_{\mathsf{b}_0}(K) \times W_{E_\mu}$.

---

[8] Note that our $\mathrm{R}\Gamma_c(\mathsf{G}^*, \mathsf{b}, \mathsf{b}_0, \mu)$ agrees with $\mathrm{R}\Gamma_c(\mathcal{M}_{\mathsf{G}^*_{\mathsf{b}_0},\mathsf{b}_0,\{\mu^{-1}\},\infty}, \Lambda)$ defined in [DvHKZ24, §8.5.8], by [DvHKZ24, Proposition 8.5.9]; but differs from $\mathrm{R}\Gamma_c(\mathsf{G}^*_{\mathsf{b}_0}, \mathsf{b}, \mu)$ defined in [Ham25, Definition 3.2] by a Tate twist and a t-shift.

Finally, we recall the following crucial result relating the cohomology of local shtuka spaces and Fargues–Scholze L-parameters. Before that, recall that for any dominant cocharacters μ for G with reflex field E_μ, the highest weight tilting module $\mathcal{T}_\mu \in \mathrm{Rep}_\Lambda(\widehat{\mathsf{G}})$ extends naturally to a representation of $\widehat{\mathsf{G}} \rtimes W_{E_\mu}$, and we define the extended highest weight tilting module

$$^{L}\mathcal{T}_\mu := \mathrm{Ind}_{\widehat{\mathsf{G}} \rtimes W_{E_\mu}}^{^L\mathsf{G}} \mathcal{T}_\mu \in \mathrm{Rep}_\Lambda(^L\mathsf{G}). \tag{3.6}$$

The isomorphism class of $^L\mathcal{T}_\mu$ only depends on the Gal_K-orbits of μ; see [Ham24, §9.1]. Moreover, we recall that the tilting module $\mathcal{T}_\mu$ equals the usual highest weight module (defined by unnormalized parabolic induction) when μ is minuscule.

Proposition 3.2.3. *Suppose $\rho \in \Pi(\mathsf{G}^*_{\mathsf{b}}, \Lambda), \pi \in \Pi(\mathsf{G}^*_{\mathsf{b}_0}, \Lambda)$ and suppose that π appears in $\mathrm{Mant}_{\mathsf{G}^*,\mathsf{b},\mathsf{b}_0,\mu}(\rho)$. Here we omit the action of W_{E_μ} on $\mathrm{Mant}_{\mathsf{G}^*,\mathsf{b},\mathsf{b}_0,\mu}(\rho)$. We then have $\phi_\pi^{\mathrm{FS}} = \phi_\rho^{\mathrm{FS}} \in \Phi^{\mathrm{ss}}(\mathsf{G}^*, \Lambda)$.*

Proof. This is established in [Ham25, Corollary 3.15]. Strictly speaking, the cited result only proves the assertion when μ is minuscule. However, the general proof is the same and we reproduce here.

Regard ρ as an element of $\mathrm{D}(\mathsf{G}^*_{\mathsf{b}}, \Lambda) \cong \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}^{\mathsf{b}}_{\mathsf{G}^*}, \Lambda)$, then there is an isomorphism

$$\mathrm{R}\Gamma_c(\mathsf{G}^*, \mathsf{b}, \mathsf{b}_0, \mu)[\rho] \cong i^*_{\mathsf{b}_0} \mathrm{T}_\mu i_{\mathsf{b},!}(\rho) \in \mathrm{D}(\mathsf{G}^*_{\mathsf{b}_0}, \Lambda)^{\mathrm{B}W_{E_\mu}},$$

by [FS24, §IX.3]. Here T_μ is the Hecke operator associated with the extended highest weight tilting module $^L\mathcal{T}_\mu$ associated with μ (3.6). Then each Schur irreducible subquotient $A \in \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_{\mathsf{G}^*}, \Lambda)$ of $\mathrm{T}_\mu i_{\mathsf{b},!}(\rho)$ has Fargues–Scholze parameter ϕ_ρ^{FS} because the excursion algebra commutes with Hecke operators. Here we omit the W_{E_μ}-action. Furthermore, each $\pi \in \Pi(\mathsf{G}^*_{\mathsf{b}_0}, \Lambda) \subset \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}^{\mathsf{b}_0}_{\mathsf{G}^*}, \Lambda)$ appearing in $i^*_{\mathsf{b}_0}(A)$ has Fargues–Scholze parameter equal to that of A under the identification $\Phi^{\mathrm{ss}}(\mathsf{G}^*_{\mathsf{b}_0}, \Lambda) = \Phi^{\mathrm{ss}}(\mathsf{G}^*, \Lambda)$, by [FS24, §IX.7.1] (see also [Ham25, Proposition 3.14]). Thus the assertion follows. ☐

3.3. **Spectral actions.** In this subsection, we assume G is quasisplit. We recall the spectral actions on sheaves on $\mathrm{Bun}_{\mathsf{G}}$ by sheaves on the stack of Langlands parameters; cf. [FS24, §X], [Ham25].

For $\phi \in \Phi^{\mathrm{ss}}(\mathsf{G}, \Lambda)$, we define $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_{\mathsf{G}}, \Lambda)_\phi \subset \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_{\mathsf{G}}, \Lambda)$ to be the full subcategory of objects A such that the endomorphism induced by any $f \in \mathcal{O}_{\mathfrak{X}_{\widehat{\mathsf{G}}}} \smallsetminus \mathfrak{m}_\phi$ is an isomorphism on A.

We recall the Act-functors defined in [Ham25, §3.2] and [MHN24, §2.3.3] via the spectral action of the moduli stack of L-parameters: Let $\mathfrak{X}_{\widehat{\mathsf{G}}} := [Z^1(W_K, \widehat{\mathsf{G}})_{\overline{\mathbb{Q}_\ell}}/\widehat{\mathsf{G}}]$ be the moduli stack of semisimple Langlands parameters defined in [DHKM24, Zhu21] and [FS24, Theorem VIII.1.3], and let $\mathcal{P}\mathrm{erf}(\mathfrak{X}_{\widehat{\mathsf{G}}})$ be the derived category of perfect complexes on $\mathfrak{X}_{\widehat{\mathsf{G}}}$. Let $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_{\mathsf{G}}, \overline{\mathbb{Q}_\ell})^\omega \subset \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_{\mathsf{G}}, \overline{\mathbb{Q}_\ell})$ be the triangulated sub-category consisting of compact objects. Then it follows from [FS24, Corollary X.1.3] that for each finite index set I, there exists a $\overline{\mathbb{Q}_\ell}$-linear action

$$\mathcal{P}\mathrm{erf}(\mathfrak{X}_{\widehat{\mathsf{G}}})^{\mathrm{B}W_K^I} \to \mathrm{End}\left(\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_{\mathsf{G}}, \overline{\mathbb{Q}_\ell})^\omega\right)^{\mathrm{B}W_K^I} : C \mapsto \{A \mapsto C \star A\},$$

which is monoidal in the sense that there exists a natural equivalence of functors:

$$(C_1 \otimes^{\mathrm{L}} C_2) \star (-) \cong C_1 \star (C_2 \star (-)).$$

Fix for the rest of this subsection a supercuspidal L-parameter $\phi \in \Phi^{\mathrm{sc}}(\mathsf{G}, \overline{\mathbb{Q}_\ell})$, then there exists a connected component C_ϕ of $\mathfrak{X}_{\widehat{\mathsf{G}}}$ consisting of unramified twists of the parameter ϕ, equipped with a natural map $C_\phi \to [\mathrm{Spd}(\overline{\mathbb{Q}_\ell})/\mathfrak{S}_\phi]$; see [FS24, §X.2]. We then have a direct summand

$$\mathcal{P}\mathrm{erf}(C_\phi) \subset \mathcal{P}\mathrm{erf}(\mathfrak{X}_{\widehat{\mathsf{G}}}),$$

and the spectral action gives rise to a direct summand

$$\mathrm{D}^{C_\phi}_{\mathrm{lis}}(\mathrm{Bun}_{\mathsf{G}}, \overline{\mathbb{Q}_\ell})^\omega \subset \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_{\mathsf{G}}, \overline{\mathbb{Q}_\ell})^\omega.$$

Then there exists a decomposition [Ham25, p. 34]

$$\mathrm{D}^{C_\phi}_{\mathrm{lis}}(\mathrm{Bun}_{\mathsf{G}}, \overline{\mathbb{Q}_\ell})^\omega \cong \bigoplus_{\mathsf{b}\in B(\mathsf{G})_{\mathrm{bas}}} \mathrm{D}^{C_\phi}(\mathsf{G}_{\mathsf{b}}, \overline{\mathbb{Q}_\ell})^\omega,$$

where $\mathrm{D}^{C_\phi}(\mathsf{G}_{\mathsf{b}}, \overline{\mathbb{Q}_\ell})^\omega \subset \mathrm{D}(\mathsf{G}_{\mathsf{b}}, \overline{\mathbb{Q}_\ell})^\omega$ is a full subcategory of the subcategory of compact objects in $\mathrm{D}(\mathsf{G}_{\mathsf{b}}, \overline{\mathbb{Q}_\ell})$. Let χ be the character of $Z(\mathsf{G})(K)$ determined by ϕ as in [Bor79, §10.1], then for each $\mathsf{b} \in B(\mathsf{G})$, the subcategory

$$\mathrm{D}^{C_\phi,\chi}(\mathsf{G}_{\mathsf{b}}, \overline{\mathbb{Q}_\ell})^\omega \subset \mathrm{D}^{C_\phi}(\mathsf{G}_{\mathsf{b}}, \overline{\mathbb{Q}_\ell})^\omega$$

spanned by the compact objects with fixed central character χ of $Z(\mathsf{G}_{\mathsf{b}}) \cong Z(\mathsf{G})$ is semisimple because supercuspidal representations are both injective and projective in the category of smooth representations with fixed central character. Thus we can identify $\mathrm{D}^{C_\phi,\chi}(\mathsf{G}_{\mathsf{b}}, \overline{\mathbb{Q}_\ell})^\omega$ with

$$\bigoplus_{\pi_{\mathsf{b}}\in\Pi_\phi(\mathsf{G}_{\mathsf{b}})} \iota_\ell\pi_{\mathsf{b}} \otimes \mathcal{P}\mathrm{erf}(\overline{\mathbb{Q}_\ell}).$$

This category is preserved under the spectral action of $\mathcal{P}\mathrm{erf}(C_\phi)$; see [Ham25, p. 34].

We now recall the Act-functors: For each $\eta \in \mathrm{Irr}(\mathfrak{S}_\phi)$, we get a line bundle $\mathcal{L}_\eta$ on C_ϕ by pulling back along the natural map $C_\phi \to [\mathrm{Spd}(\overline{\mathbb{Q}_\ell})/\mathfrak{S}_\phi]$, and we define Act_η to be the spectral action of this line bundle on $\mathrm{D}^{C_\phi,\chi}(\mathsf{G}_{\mathsf{b}}, \overline{\mathbb{Q}_\ell})^\omega$. In particular, Act-functors are symmetric monoidal, i.e., $\mathrm{Act}_{\mathbf{1}}$ is the identity functor, and for any $\eta, \eta' \in \mathrm{Irr}(\mathfrak{S}_\phi)$, there exists a natural equivalence of functors

$$\mathrm{Act}_\eta \circ \mathrm{Act}_{\eta'} \cong \mathrm{Act}_{\eta\eta'}.$$

From this it is easy to show that each Act-functor sends an irreducible admissible representation

$$\iota_\ell\pi \in \bigcup_{\mathsf{b}\in B(\mathsf{G})_{\mathrm{bas}}} \iota_\ell\Pi_\phi(\mathsf{G}_{\mathsf{b}})$$

to another irreducible admissible representation

$$\iota_\ell\pi' \in \bigcup_{\mathsf{b}\in B(\mathsf{G})_{\mathrm{bas}}} \iota_\ell\Pi_\phi(\mathsf{G}_{\mathsf{b}})$$

with a t-shift; see [MHN24, Lemma 2.28].

3.4. **Weak version of the Kottwitz conjecture.** In this subsection, let G^* be a special orthogonal or unitary group as defined in §2.1, and let $(G = G^*_{b_0}, \varrho_{b_0}, z_{b_0})$ be the pure inner twist of G^* associated with $b_0 \in B(G^*)_{\mathrm{bas}}$. Let μ be a dominant cocharacter of $G^*_{\overline{K}}$, and let $b \in B(G^*, b_0, \mu)_{\mathrm{bas}}$ be the unique nontrivial basic element, regarded as a basic element in $B(G)_{\mathrm{bas}}$ via the isomorphism (2.2), so we can adopt the notation from §3.2. Thus $G_b \cong G^*_{b_0+b}$, and we define

$$\mathrm{Mant}_{G,b,\mu} := \mathrm{Mant}_{G^*,b_0+b,b_0,\mu} : \mathrm{K}_0(G_b, \overline{\mathbb{Q}_\ell}) \to \mathrm{K}_0(G, \overline{\mathbb{Q}_\ell}),$$

after forgetting the W_{E_μ}-action.

We will use the weak version of the Kottwitz conjecture from [HKW22] describing the cohomology $\mathrm{Mant}_{G,b,\mu}(\iota_\ell\rho)$ for $\rho \in \Pi_\phi(G_b)$, where ϕ is a discrete L-parameter. The (generalized) Kottwitz conjecture describes $\mathrm{Mant}_{G,b,\mu}(\iota_\ell\rho)$ in terms of the local Langlands correspondence, where ρ lies in a supercuspidal L-packet.

In [HKW22, Theorem 1.0.2], a weak version of the Kottwitz conjecture is established for all discrete L-parameters, but disregarding the action of the Weil group, and modulo a virtual representation whose character vanishes on the locus of elliptic elements. Their proof is conditional on the refined local Langlands conjecture of [Kal16, Conjecture G] (in fact, as G is always a pure inner form of G^*, the isocrystal version [Kal16, Conjecture G] suffices), but in Case O2 (with sufficiently high rank), only the weak version of this conjecture stated in §2 is known. To remedy this, we will use weak versions of the endoscopic character identities Theorem 2.3.4 to prove a weak version of [HKW22, Theorem 1.0.2].

To state results uniformly, for every $b \in B(G)_{\mathrm{bas}}$, define

$$\tilde{\mathrm{K}}_0(G_b, \overline{\mathbb{Q}_\ell}) := \mathrm{K}_0(G_b, \overline{\mathbb{Q}_\ell}) / \left\langle [\pi] - [\pi^\varsigma] : \pi \in \Pi(G_b, \overline{\mathbb{Q}_\ell}) \right\rangle.$$

The natural ς-action on the local shtuka space induces a well-defined map

$$\mathrm{Mant}_{G,b,\mu} : \tilde{\mathrm{K}}_0(G_b, \overline{\mathbb{Q}_\ell}) \to \tilde{\mathrm{K}}_0(G, \overline{\mathbb{Q}_\ell}).$$

The set of elliptic elements of $G_b(K)$ is invariant under the action of ς, so it makes sense to talk about an object of $\tilde{\mathrm{K}}_0(G_b, \overline{\mathbb{Q}_\ell})$ whose character vanishes on the locus of elliptic elements of $G_b(K)$, and these objects are exactly those coming from a proper Levi subgroup of G_b [HKW22, Theorem C.1.1].

Similarly, for $\tilde{\pi} \in \tilde{\Pi}(G_b)$, we can define the Harish-Chandra character

$$\Theta_{\tilde{\pi}} = \frac{1}{2}(\Theta_\pi + \Theta_{\pi^\varsigma}) \in C(G_b(K)_{\mathrm{s.reg}} /\!\!/ G_b(K), \mathbb{C}),$$

where $\pi \in \Pi(G_b)$ is an arbitrary representative of $\tilde{\pi}$.

For $\tilde{\rho} \in \tilde{\Pi}(G_b)$, the Fargues–Scholze parameter $\iota_\ell^{-1}\phi_{\tilde{\rho}}^{\mathrm{FS}} : W_K \to {}^LG^*$ is well-defined up to $\mathrm{O}_{N(G)}(\mathbb{C})$-conjugation in Case O2, by the compatibility of Fargues–Scholze LLC with central extensions, Theorem 3.1.1, so it is unambiguous to ask whether it is a supercuspidal L-parameter.

Then our theorem, which slightly generalizes the main theorem of [HKW22], is stated as follows (a stronger version will be established in §7.3):

Theorem 3.4.1. *If $\tilde{\phi} \in \tilde{\Phi}_2(G^*)$ is a discrete L-parameter, and $\tilde{\rho} \in \tilde{\Pi}_{\tilde{\phi}}(G_b)$, then*

$$\mathrm{Mant}_{G,b,\mu}(\iota_\ell\tilde{\rho}) = \sum_{\tilde{\pi} \in \tilde{\Pi}_{\tilde{\phi}}(G)} \dim \mathrm{Hom}_{\mathfrak{S}_{\tilde{\phi}}}(\delta[\tilde{\pi}, \tilde{\rho}], \mathcal{T}_\mu)[\iota_\ell\tilde{\pi}] + \mathrm{Err}$$

in $\tilde{\mathrm{K}}_0(G, \overline{\mathbb{Q}_\ell})$, where $\mathrm{Err} \in \tilde{\mathrm{K}}_0(G, \overline{\mathbb{Q}_\ell})$ is a virtual representation whose character vanishes on $G(K)_{\mathrm{s.reg,ell}}$.

Moreover, if the packet $\tilde{\Pi}_{\tilde{\phi}}(G)$ consists entirely of supercuspidal representations and the Fargues–Scholze L-parameter $\tilde{\phi}_{\tilde{\rho}}^{\mathrm{FS}}$ is supercuspidal, then $\mathrm{Err} = 0$.

We can apply Theorem 3.4.1 to $(b, \mu) = (b_1, \mu_1)$ defined in §2.6. Then it follows from Theorem 2.6.1 that we obtain the following corollary:

Corollary 3.4.2. *Suppose $\tilde{\phi} \in \tilde{\Phi}_2(G^*)$ is a discrete L-parameter and*

$$\tilde{\phi}^{\mathrm{GL}} = \phi_1 \oplus \cdots \oplus \phi_k \oplus \phi_{k+1} \oplus \cdots \oplus \phi_r$$

where the ϕ_i are irreducible representations of $W_{K_1} \times \mathrm{SL}_2(\mathbb{C})$ of dimension d_i for each i, and d_i is odd if and only if $i \leq k$. Let μ_1 be the dominant cocharacter of $G^*_{\overline{K}}$ defined in (2.5), and $b_1 \in B(G^*, b_0, \mu_1)_{\mathrm{bas}}$ be the unique basic element. For

$$\tilde{\rho} = \tilde{\pi}_{[I]} \in \tilde{\Pi}_{\tilde{\phi}}(G_{b_1}), \qquad \#I \equiv \frac{\kappa_{b_0}(-\mathbf{1}) + 1}{2} \pmod 2,$$

one has

$$\mathrm{Mant}_{G,b_1,\mu_1}(\iota_\ell\tilde{\pi}_{[I]}) = \mathrm{Mant}_{G,b_1,\mu_1^\bullet}(\iota_\ell\tilde{\pi}_{[I]}) = \sum_{i \in [r]_+} d_i[\iota_\ell\tilde{\pi}_{[I \oplus \{i\}]}] + \mathrm{Err}$$

in $\tilde{\mathrm{K}}_0(G, \overline{\mathbb{Q}_\ell})$. Here $\mu_1^\bullet = -w_0(\mu_1)$ is the dominant cocharacter conjugate to μ_1^{-1}, and $\mathrm{Err} \in \tilde{\mathrm{K}}_0(G, \overline{\mathbb{Q}_\ell})$ is a virtual representation whose character vanishes on $G(K)_{\mathrm{s.reg,ell}}$.

Corollary 3.4.3. *If $\tilde{\phi} \in \tilde{\Phi}_{\mathrm{sc}}(G^*)$ is a supercuspidal L-parameter, then all representations $\tilde{\pi} \in \tilde{\Pi}_{\tilde{\phi}}(G)$ have the same Fargues–Scholze L-parameter $\tilde{\phi}_{\iota_\ell\tilde{\pi}}^{\mathrm{FS}}$.*

Proof. Let $I \subset [r]_+$ with $\#[I] \equiv \frac{\kappa_{b_0}(\mathbf{-1})+1}{2} \pmod 2$. For every $i \in [r]_+$, it follows from Corollary 3.4.2 that $\tilde{\pi}_{[I\oplus\{i\}]}$ appears in $\mathrm{Mant}_{G,b_1,\mu_1}(\iota_\ell \tilde{\pi}_{[I]})$. Here we use that the error term has no supercuspidal constituents. Proposition 3.2.3 therefore gives

$$\tilde{\phi}^{\mathrm{FS}}_{\iota_\ell \tilde{\pi}_{[I\oplus\{i\}]}} = \tilde{\phi}^{\mathrm{FS}}_{\iota_\ell \tilde{\pi}_{[I]}} = \tilde{\phi}^{\mathrm{FS}}_{\iota_\ell \tilde{\pi}_{[I\oplus\{j\}]}}$$

for any $i, j \in [r]_+$. This implies that $\tilde{\phi}^{\mathrm{FS}}_{\iota_\ell \tilde{\pi}_{[J]}} = \tilde{\phi}^{\mathrm{FS}}_{\iota_\ell \tilde{\pi}_{[J']}}$ for any $J \subset J' \subset [r]_+$ with $\#J' = \#J + 1$. Clearly, this implies that $\tilde{\phi}^{\mathrm{FS}}_{\iota_\ell \tilde{\pi}_{[J]}} = \tilde{\phi}^{\mathrm{FS}}_{\iota_\ell \tilde{\pi}_{[J']}}$ for any $J, J' \subset [r]_+$ with $\#J \equiv \#J' \pmod 2$. □

We first recall from [HKW22, Definition 3.2.4] the transfer map from conjugation-invariant functions on $G_b(K)_{\mathrm{s.reg}}$ to conjugation-invariant functions on $G(K)_{\mathrm{s.reg}}$ when $b \in B(G)_{\mathrm{bas}}$ is basic.

Definition 3.4.4. There is a diagram of topological spaces

$$\begin{tikzcd} & \mathrm{Rel}_b \arrow[dl] \arrow[dr] & \\ G(K)_{\mathrm{s.reg}} /\!\!/ G(K) & & G_b(K)_{\mathrm{s.reg}} /\!\!/ G_b(K), \end{tikzcd}$$

where Rel_b is the set of $G(K) \times G_b(K)$-conjugacy classes of triples (g, g', λ) such that

- $g \in G(K)_{\mathrm{s.reg}}$ and $g' \in G_b(K)_{\mathrm{s.reg}} \subset G(\breve{K})$ are stably conjugate, i.e., conjugate under the action of $G(\breve{K})$,
- $\lambda \in X_\bullet(Z_G(g))$ satisfies $\kappa_{Z_G(g)}(\mathrm{inv}[b](g, g'))$ agrees with the image of λ in $X_\bullet(Z_G(g))_{\mathrm{Gal}_K}$. Here $\mathrm{inv}[b](g, g')$ is the class of $y^{-1} b \varphi_K(y)$ in $B(Z_G(g))$, where $y \in G(\breve{K})$ satisfies $g' = ygy^{-1}$. This class is independent of the choice of y; see [HKW22, Definition 3.2.2, Fact 3.2.3].

Here $(z, z') \in G(K) \times G_b(K)$ acts by conjugation on such triples by

$$\mathrm{ad}(z, z')(g, g', \lambda) = (\mathrm{ad}(z)g, \mathrm{ad}(z')g', \mathrm{ad}(z)\lambda),$$

and Rel_b is given the subspace topology by the inclusion $\mathrm{Rel}_b \subset (G(K) \times G_b(K) \times X_\bullet(G))/(G(K) \times G_b(K))$ with $X_\bullet(G)$ being discrete.

We recall the following Hecke transfer map from [HKW22, Definition 3.2.7, Definition 6.3.4].

Definition 3.4.5. We define a Hecke transfer map

$$T^{G_b \to G}_{b,\mu} : C(G_b(K)_{\mathrm{s.reg}} /\!\!/ G_b(K)) \to C(G(K)_{\mathrm{s.reg}} /\!\!/ G(K))$$

such that

$$\left[T^{G_b \to G}_{b,\mu}(f')\right](g) = (-1)^{\langle \mu, 2\rho_{G^*}\rangle} \sum_{(g,g',\lambda) \in \mathrm{Rel}_b} f'(g') \dim \mathcal{T}_\mu[\lambda].$$

The representation $\mathcal{T}_\mu$ has only finitely many weights, and the relevant fibers of Rel_b over a fixed strongly regular element are finite. Hence the above sum is finite. The same finiteness, together with the properness property of the relation used in the definition of the transfer map, shows that $T^{G_b \to G}_{b,\mu}(f')$ is compactly supported whenever f' is compactly supported.

Moreover, on the strongly regular elliptic locus, $T^{G_b \to G}_{b,\mu}$ can be extended to a Hecke map on invariant distributions

$$\mathcal{T}^{G_b \to G}_{b,\mu} : \mathrm{Dist}(G_b(K)_{\mathrm{s.reg,ell}})^{G_b(K)} \to \mathrm{Dist}(G(K)_{\mathrm{s.reg,ell}})^{G(K)}.$$

We then have the following result of Hansen, Kaletha and Weinstein [HKW22, Theorem 6.5.2]:

Proposition 3.4.6. *For any $\rho \in \Pi(G_b)$ and $f \in C_c(G(K)_{\mathrm{s.reg,ell}})$,*

$$\mathrm{tr}\left(f | \iota_\ell^{-1} \mathrm{Mant}_{G,b,\mu}(\iota_\ell \rho)\right) = \left[\mathcal{T}^{G_b \to G}_{b,\mu}(\Theta_\rho)\right](f).$$

In particular, the virtual character of $\iota_\ell^{-1}\operatorname{Mant}_{G,b,\mu}(\iota_\ell\rho)$ *restricted to* $G(K)_{\mathrm{s.reg,ell}}$ *is equal to* $T_{b,\mu}^{G_b\to G}(\Theta_\rho)$.

We now use the weak endoscopic character identities to prove an analogue of [HKW22, Theorem 3.2.9], which relates the Hecke transfer map $T_{b,\mu}^{G_b\to G}$ to classical LLCs of G and G_b:

Proposition 3.4.7. *Assume that* $b\in B(G^*,b_0,\mu)_{\mathrm{bas}}$ *is basic, and let* $\phi\in\Phi_2(G^*)$ *be a discrete* L*-parameter. Then, for every* $\tilde\rho\in\tilde\Pi_{\tilde\phi}(G_b)$ *and every* $g\in G(K)_{\mathrm{s.reg}}$ *that transfers to* $G_b(K)$,

$$\left[T_{b,\mu}^{G_b\to G}\Theta_{\tilde\rho}\right](g)=\sum_{\tilde\pi\in\tilde\Pi_{\tilde\phi}(G)}\dim\operatorname{Hom}_{\mathfrak{S}_{\tilde\phi}}(\delta[\tilde\pi,\tilde\rho],\mathcal{T}_\mu)\Theta_{\tilde\pi}(g).$$

Proof. We modify the argument of [HKW22, §3.3]. We first clarify the three related endoscopic elements that occur in the comparison between the rigid and isocrystal normalizations. Let

$$\operatorname{pr}_\phi,\nu_\phi:S_\phi^+\longrightarrow S_\phi$$

denote, respectively, the natural projection and the comparison homomorphism defined in [Kal18, (4.7)]. Since ϕ is discrete, we identify $S_\phi^\natural=S_\phi=\mathfrak{S}_\phi$.

Choose $\dot s\in S_\phi^+$, and set

$$s=\operatorname{pr}_\phi(\dot s),\qquad s^\natural=\nu_\phi(\dot s).$$

Here $\dot s$ is the refined endoscopic element used in the rigid normalization, s is its ordinary image and determines the underlying endoscopic group, while $s^\natural$ is the corresponding element in the isocrystal normalization and is the element at which the packet characters and $\mathcal{T}_\mu$ are evaluated. By [Kal18, §4.2], the elements s and $s^\natural$ differ by an element of $Z(\widehat G)^{\mathrm{Gal}_K}$. Consequently, they have the same image

$$\overline{s}\in\overline{\mathfrak{S}}_\phi$$

and determine the same underlying endoscopic group and L-embedding.

Let

$$\mathfrak{e}=(G^{\mathfrak{e}},s^\natural,{}^L\xi^{\mathfrak{e}})$$

be the extended endoscopic triple associated with $s^\natural$, and let $\phi^{\mathfrak{e}}\in\Phi_2(G^{\mathfrak{e}})$ be the parameter associated with ϕ. Its rigid refinement

$$\dot{\mathfrak{e}}=(G^{\mathfrak{e}},{}^LG^{\mathfrak{e}},\dot s,{}^L\xi^{\mathfrak{e}})$$

is the refined endoscopic datum attached to $\dot s$ in [HKW22, (A.1.1)].

For every $(g,g',\lambda)\in\mathrm{Rel}_b$, we have

$$\begin{aligned}
&e(G_b)\sum_{\tilde\rho'\in\tilde\Pi_{\tilde\phi}(G_b)}\iota_{\mathfrak{w},b_0+b}(\tilde\rho')(s^\natural)\Theta_{\tilde\rho'}(g')\\
&\overset{\text{Theorem 2.3.4}}{=}\sum_{h\in G^{\mathfrak{e}}(K)_{\mathrm{s.reg}}/\mathrm{st.conj}}\Delta[\mathfrak{w},\mathfrak{e},z_{b_0+b}](h,g')S\Theta_{\tilde\phi^{\mathfrak{e}}}(h)\\
&\overset{\text{[HKW22, Lemma A.1.1]}}{=}\sum_{h\in G^{\mathfrak{e}}(K)_{\mathrm{s.reg}}/\mathrm{st.conj}}\Delta[\mathfrak{w},\mathfrak{e},z_{b_0}](h,g)\left\langle\operatorname{inv}[b](g,g'),s_{h,g}^\natural\right\rangle S\Theta_{\tilde\phi^{\mathfrak{e}}}(h)\\
&=\sum_{h\in G^{\mathfrak{e}}(K)_{\mathrm{s.reg}}/\mathrm{st.conj}}\Delta[\mathfrak{w},\mathfrak{e},z_{b_0}](h,g)\lambda(s_{h,g}^\natural)S\Theta_{\tilde\phi^{\mathfrak{e}}}(h).
\end{aligned}$$

Here $s_{h,g}^\natural$ is obtained by transporting $s^\natural$ to

$$\widehat{Z_G(g)}^{\mathrm{Gal}_K}$$

through the admissible isomorphism between the centralizers of h and g. It is this transported element that is paired with $\operatorname{inv}[b](g,g')$ in [HKW22, Lemma A.1.1].

We now multiply this expression by the kernel function $\dim \mathcal{T}_\mu[\lambda]$, and sum over all $g' \in G_b(K)_{\mathrm{s.reg}} /\!\!/ G_b(K)$ and $\lambda \in X_\bullet(Z_G(g))$ such that $(g, g', \lambda) \in \mathrm{Rel}_b$.

$$\begin{aligned}
& e(G_b) \sum_{(g',\lambda)} \sum_{\tilde{\rho}' \in \tilde{\Pi}_{\tilde{\phi}}(G_b)} \iota_{\mathfrak{w}, b_0+b}(\tilde{\rho}')(s^\natural)\Theta_{\tilde{\rho}'}(g') \dim \mathcal{T}_\mu[\lambda] \\
= & \sum_{h \in G^{\mathfrak{e}}(K)_{\mathrm{s.reg}}/\mathrm{st.conj}} \Delta[\mathfrak{w}, \mathfrak{e}, z_{b_0}](h, g) \mathrm{S}\Theta_{\tilde{\phi}^{\mathfrak{e}}}(h) \sum_{(g',\lambda)} \lambda(s^\natural_{h,g}) \dim \mathcal{T}_\mu[\lambda] \\
= & \sum_{h \in G^{\mathfrak{e}}(K)_{\mathrm{s.reg}}/\mathrm{st.conj}} \Delta[\mathfrak{w}, \mathfrak{e}, z_{b_0}](h, g) \mathrm{S}\Theta_{\tilde{\phi}^{\mathfrak{e}}}(h) \mathrm{tr}(\mathcal{T}_\mu(s^\natural_{h,g})) \\
= & \mathrm{tr}(\mathcal{T}_\mu(s^\natural)) \sum_{h \in G^{\mathfrak{e}}(K)_{\mathrm{s.reg}}/\mathrm{st.conj}} \Delta[\mathfrak{w}, \mathfrak{e}, z_{b_0}](h, g) \mathrm{S}\Theta_{\tilde{\phi}^{\mathfrak{e}}}(h) \\
\overset{\text{Theorem 2.3.4}}{=} & \mathrm{tr}(\mathcal{T}_\mu(s^\natural)) e(G) \sum_{\tilde{\pi} \in \tilde{\Pi}_{\tilde{\phi}}(G)} \iota_{\mathfrak{w}, b_0}(\tilde{\pi})(s^\natural)\Theta_{\tilde{\pi}}(g).
\end{aligned}$$

Here the second and third equations hold by the argument of [HKW22, p. 17].

We multiply the above equation by $\iota_{\mathfrak{w}, b_0+b}(\tilde{\rho})(s^\natural)^{-1}$, then as a function of $s^\natural \in \mathfrak{S}_{\tilde{\phi}}$, the left-hand side is invariant under translation by z_ϕ, so the same holds for the right-hand side, and both sides become functions of $\overline{s} \in \overline{\mathfrak{S}}_\phi$. We now average over $\overline{s} \in \overline{\mathfrak{S}}_\phi$ to get

$$\begin{aligned}
& \frac{1}{\#\overline{\mathfrak{S}}_\phi} e(G_b) \sum_{\overline{s} \in \overline{\mathfrak{S}}_\phi} \sum_{(g',\lambda)} \sum_{\tilde{\rho}' \in \tilde{\Pi}_{\tilde{\phi}}(G_b)} \frac{\iota_{\mathfrak{w}, b_0+b}(\tilde{\rho}')}{\iota_{\mathfrak{w}, b_0+b}(\tilde{\rho})}(s^\natural)\Theta_{\tilde{\rho}'}(g') \dim \mathcal{T}_\mu[\lambda] \\
= & \frac{1}{\#\overline{\mathfrak{S}}_\phi} e(G) \sum_{\overline{s} \in \overline{\mathfrak{S}}_\phi} \mathrm{tr}(\mathcal{T}_\mu(s^\natural)) \sum_{\tilde{\pi} \in \tilde{\Pi}_{\tilde{\phi}}(G)} \frac{\iota_{\mathfrak{w}, b_0}(\tilde{\pi})}{\iota_{\mathfrak{w}, b_0+b}(\tilde{\rho})}(s^\natural)\Theta_{\tilde{\pi}}(g),
\end{aligned}$$

where we recall that $s^\natural \in \mathfrak{S}_\phi$ is a lift of $\overline{s}$. By Fourier inversion, the left-hand side equals

$$e(G_b) \sum_{(g',\lambda)} \Theta_{\tilde{\rho}}(g') \dim \mathcal{T}_\mu[\lambda] = (-1)^{\langle \mu, 2\rho_{G^*} \rangle} e(G_b) \left[T^{G_b \to G}_{b,\mu} \Theta_{\tilde{\rho}} \right](g),$$

and the right-hand side equals

$$e(G) \sum_{\tilde{\pi} \in \tilde{\Pi}_{\tilde{\phi}}(G)} \Theta_{\tilde{\pi}}(g) \frac{1}{\#\overline{\mathfrak{S}}_{\tilde{\phi}}} \sum_{\overline{s} \in \overline{\mathfrak{S}}_{\tilde{\phi}}} \mathrm{tr}(\mathcal{T}_\mu(s^\natural))\delta[\tilde{\pi}, \tilde{\rho}]^{-1}(s^\natural) = e(G) \sum_{\tilde{\pi} \in \tilde{\Pi}_{\tilde{\phi}}(G)} \dim \mathrm{Hom}_{\mathfrak{S}_{\tilde{\phi}}}(\delta[\tilde{\pi}, \tilde{\rho}], \mathcal{T}_\mu)\Theta_{\tilde{\pi}}(g).$$

So the assertion is reduced to the identity $e(G) = (-1)^{\langle \mu, 2\rho_{G^*} \rangle} e(G_b)$, which is exactly [HKW22, (3.3.3)]. ☐

Now Theorem 3.4.1 follows from the above propositions in the same way as in the proof of [HKW22, Theorem 6.5.1].

Proof of Theorem 3.4.1. The equality in $\tilde{\mathrm{K}}_0(G(K))$ follows from Propositions 3.4.6 and 3.4.7.

It remains to prove the assertion about Err. Choose a representative $\rho \in \Pi(G_b)$ of $\tilde{\rho}$. Let $\pi \in \Pi(G)$ have nonzero coefficient in

$$\iota_\ell^{-1} \mathrm{Mant}_{G,b,\mu}(\iota_\ell \rho).$$

By Proposition 3.2.3,

$$\phi^{\mathrm{FS}}_{\iota_\ell \pi} = \phi^{\mathrm{FS}}_{\iota_\ell \rho}.$$

The parameter on the right is supercuspidal by assumption. Suppose that π were not supercuspidal. By the theory of cuspidal support, π would be an irreducible subquotient of a normalized parabolic induction from a proper Levi subgroup of G. Compatibility of the Fargues–Scholze correspondence with parabolic induction, as stated in Theorem 3.1.1, would then imply that $\phi^{\mathrm{FS}}_{\iota_\ell\pi}$ factors through the L-group of a proper Levi subgroup. This contradicts its supercuspidality. Therefore

$$\mathrm{Mant}_{G,b,\mu}(\iota_\ell\rho)$$

is a virtual sum of supercuspidal representations. The same is true of the explicit packet term in the formula, so Err is a virtual sum of supercuspidal representations.

On the other hand, the character of Err vanishes on the strongly regular elliptic locus. By [HKW22, Theorem C.1.1], its class is generated by normalized parabolic inductions from proper Levi subgroups. Such induced representations have no supercuspidal irreducible constituents. Since irreducible classes form a basis of the Grothendieck group, the subgroups generated respectively by supercuspidal and non-supercuspidal irreducible representations have zero intersection. Hence $\mathrm{Err} = 0$. ☐

4. Cohomology of orthogonal and unitary Shimura varieties

In this section, we compute the Π-isotypic component of the cohomology of Shimura varieties $\mathrm{Sh}(\mathbf{G}, \mathbf{X})$ of orthogonal or unitary type related to the local group G defined in §2.1, where Π is a special cuspidal automorphic representation of $\mathbf{G}(\mathbf{A}_f)$. These results are related to the cohomology of local shtuka spaces via the basic uniformization theorem stated in the next section, §5.

Let F be a totally real number field, and fix a nontrivial additive character ψ_F of $F\backslash\mathbf{A}_F$, which extends to an additive character of any finite extension F'/F by defining $\psi_{F'} := \psi_F \circ \mathrm{tr}_{F'/F}$. Let F_1 be either F or a CM field containing F, and let $\mathsf{c} \in \mathrm{Gal}(F_1/F)$ be the element with fixed field F. Let $\tau_0 : F_1 \to \mathbb{C}$ be a fixed embedding. Denote by $\chi_{F_1/F} : \mathbf{A}_F^\times/F^\times \to \{\pm 1\}$ the character associated with F_1/F via global class field theory. If $F_1 \neq F$, choose a totally imaginary element $\daleth \in F_1^\times$, so each embedding $\tau : F \to \mathbb{C}$ extends to an embedding $\tau : F_1 \to \mathbb{C}$ sending $\daleth$ to $\mathbb{R}_+\mathrm{i}$.

4.1. **The groups.** Let $\mathbf{V}$ be a vector space over F_1 equipped with a nondegenerate Hermitian c-sesquilinear form $\langle -, - \rangle$ on $\mathbf{V}$, i.e.

$$\langle au + bv, w \rangle = a\,\langle u, w \rangle + b\,\langle v, w \rangle\,,$$

$$\langle v, w \rangle = \langle w, v \rangle^{\mathsf{c}}\,.$$

In Case O2, choose an arbitrary diagonal basis $\{v_1, \ldots, v_{\dim(\mathbf{V})}\}$ of $\mathbf{V}$ over F_1 such that $\langle v_i, v_i \rangle = a_i \in F^\times$. Define

$$\mathrm{disc}(\mathbf{V}) = (-1)^{\binom{\dim(\mathbf{V})}{2}} 2^{-\dim(\mathbf{V})} \prod_{i=1}^{\dim(\mathbf{V})} a_i$$

whose image in $F^\times/(F^\times)^2$ is independent of the basis chosen, and we write $\mathrm{disc}(\mathbf{G}) = \mathrm{disc}(\mathbf{V})$.

Let $\mathrm{U}(\mathbf{V}) \leq \mathrm{GL}(\mathbf{V})$ be the algebraic subgroup defined by

$$\mathrm{U}(\mathbf{V}) = \{g \in \mathrm{GL}(\mathbf{V}) : \langle gv, gw \rangle = \langle v, w \rangle\, \forall v, w \in \mathbf{V}\},$$

and let $\mathbf{G} = \mathrm{U}(\mathbf{V})^\circ$ be the neutral component of $\mathrm{U}(\mathbf{V})$. Let $\mathbf{G}^*$ be the unique quasisplit inner form of $\mathbf{G}$ over F. Exactly one of the following cases occurs:

- **O1** If $F_1 = F$ and $\dim(\mathbf{V}) = 2n+1$ is odd, then $\mathbf{G}^* = \mathrm{SO}_{2n+1}$.
- **O2** If $F_1 = F$ and $\dim(\mathbf{V}) = 2n$ is even, then $\mathbf{G}^* = \mathrm{SO}_{2n}^{\mathrm{disc}(\mathbf{V})}$ is the special orthogonal group associated with the quadratic space $\mathbf{V}^*$ over F of dimension $2n$, discriminant $\mathrm{disc}(\mathbf{V})$ such that the Hasse–Witt invariants of $\mathbf{V}^* \otimes F_v$ are 1 for each $v \in \Sigma_F$.
- **U** If $F_1 \neq F$ and $\dim(\mathbf{V}) = n$, then $\mathbf{G}^* = \mathrm{U}(n)$ is the unitary group associated with the Hermitian space $\mathbf{V}^*$ of dimension n with respect to the quadratic extension F_1/F such that the Hasse–Witt invariants of $\mathbf{V}^* \otimes F_v$ are 1 for each $v \in \Sigma_F$.

We refer to cases O1 and O2 together as Case O. We assume further that

$$\dim \mathbf{V} \geq \begin{cases} 2 & \text{in Case U} \\ 5 & \text{in Case O1} \\ 6 & \text{in Case O2} \end{cases},$$

so that $\mathbf{G}_{\mathrm{ad}}$ is always geometrically simple.

To unify notation, we write $n(\mathbf{G}) = n(\mathbf{G}^*)$ for the rank of $\mathbf{G}_{\overline{F}}$, define $N(\mathbf{G}), d(\mathbf{G}), b(\mathbf{G})$ analogous to the local case (2.1), and define $\mathrm{disc}(\mathbf{G}) := \mathrm{disc}(\mathbf{V})$ in Case O2.

We fix a pinning $(\mathbf{B}^*, \mathbf{T}^*, \{X_\alpha^*\}_{\alpha\in\Delta})$ of $\mathbf{G}^*$ by identifying $\mathbf{G}^* = \mathrm{U}(\mathbf{V}^*)^\circ$ for a suitable c-Hermitian space $\mathbf{V}^*$ over F_1, and choosing a complete flag of totally isotropic subspaces in $\mathbf{V}^*$. Together with the additive character ψ_F, this pinning determines a Whittaker datum $\mathfrak{w}$ for $\mathbf{G}^*$.

It follows from the theorem of Hasse–Minkowski and Landherr [Gro21, Theorem 2.1, Theorem 3.1] and [GGP12, Lemma 2.1] that pure inner twists of $\mathbf{G}^*$ are in bijection with isometry classes of c-Hermitian spaces $\mathbf{V}$ with respect to F_1/F of dimension $d(\mathbf{G}^*)$ (and also with discriminant $\mathrm{disc}(\mathbf{V})$ in Case O2), and these isometry classes are determined by isometry classes of localizations $\mathbf{V}_v$ for each $v \in \Sigma_F$. In particular, $\mathbf{G}$ can always be realized as a pure inner twist $(\mathbf{G}, \boldsymbol{\varrho}, \boldsymbol{z})$ of $\mathbf{G}^*$.

We fix an isomorphism

$$\widehat{\mathbf{G}} \cong \begin{cases} \mathrm{Sp}_{N(\mathbf{G})}(\mathbb{C}) & \text{in Case O1} \\ \mathrm{SO}_{N(\mathbf{G})}(\mathbb{C}) & \text{in Case O2} \\ \mathrm{GL}_{N(\mathbf{G})}(\mathbb{C}) & \text{in Case U} \end{cases},$$

and fix a pinning $(\widehat{\mathbf{T}}, \widehat{\mathbf{B}}, \{X_\alpha\}_{\alpha\in\Delta})$ where $\widehat{\mathbf{T}}$ is the diagonal torus, $\widehat{\mathbf{B}}$ is the group of upper triangular matrices, and $\{X_\alpha\}_{\alpha\in\Delta}$ is the set of standard root vectors. We write ${}^L\mathbf{G} = \widehat{\mathbf{G}} \rtimes W_F$ for the Langlands L-group in the Weil form. Note that $\widehat{\mathbf{G}}$ has a standard representation $\widehat{\mathrm{Std}} = \widehat{\mathrm{Std}}_{\mathbf{G}} : \widehat{\mathbf{G}} \to \mathrm{GL}_{N(\mathbf{G}),\mathbb{C}}$.

As in the local case §2.1, there exists an automorphism θ of $\mathbf{G}^{\mathrm{GL}} := \mathrm{Res}_{F_1/F}\mathrm{GL}_{N(\mathbf{G})}$ such that $\mathbf{G}^*$ determines an element in $\mathcal{E}_{\mathrm{ell}}(\mathbf{G}^{\mathrm{GL}} \rtimes \theta)$, and the description of the isomorphism classes of elliptic endoscopic triples $\mathfrak{e} \in \mathcal{E}_{\mathrm{ell}}(\mathbf{G})$ is similar to the local case.

Finally, we define a central extension $\mathbf{G}^\sharp$ of $\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G}$ as follows:

- In Case O, we imitate [Car86, p. 163]. Let $\mathrm{Cl}(\mathbf{V})$ and $\mathrm{Cl}^\circ(\mathbf{V})$ be the Clifford algebra and even Clifford algebra, respectively. Note that there exists an embedding $\mathbf{V} \subset \mathrm{Cl}(\mathbf{V})$ and an anti-involution $*$ on $\mathrm{Cl}(\mathbf{V})$ (the main involution) [MP16, §1.1]. Let $\mathrm{GSpin}(\mathbf{V})$ be the stabilizer in $\mathrm{Cl}^\circ(\mathbf{V})^\times$ of $\mathbf{V} \subset \mathrm{Cl}(\mathbf{V})$ with respect to the conjugation action of $\mathrm{Cl}^\circ(\mathbf{V})^\times$ on $\mathrm{Cl}(\mathbf{V})$, which is a reductive group over F. The conjugation action of $\mathrm{GSpin}(\mathbf{V})$ on $\mathbf{V}$ induces an exact sequence of reductive groups over F:

(4.1) $$1 \to \mathrm{GL}_{1,F} \to \mathrm{GSpin}(\mathbf{V}) \to \mathbf{G} \to 1.$$

  There is a similitude map $\nu : \mathrm{GSpin}(\mathbf{V}) \to \mathrm{GL}_{1,F} : g \mapsto g^* g$ whose restriction on the central torus is $z \mapsto z^2$. The kernel $\mathrm{Spin}(\mathbf{V}) := \ker(\nu)$ is called the spinor group of $\mathbf{V}$.

  Fix an imaginary quadratic element $\daleth \in \mathbb{R}_+\mathrm{i}$ (in particular $\daleth^2 \in \mathbb{Q}_-$). Define

$$\mathbf{G}_\daleth = \left((\mathrm{Res}_{F/\mathbb{Q}}\,\mathrm{GSpin}(\mathbf{V})) \times \mathrm{Res}_{F(\daleth)/\mathbb{Q}}\mathrm{GL}_1\right)/\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_1,$$

  where $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_1$ is embedded anti-diagonally. Define

$$\nu^\sharp : \mathbf{G}_\daleth \to \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_1 : (g,t) \mapsto \nu(g)\,\mathrm{Nm}_{F(\daleth)/F}(t),$$

  and we define $\mathbf{G}^\sharp \subset \mathbf{G}_\daleth$ to be the inverse image of the subtorus $\mathrm{GL}_1 \subset \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_1$ under $\nu^\sharp$. Then

$$[\mathbf{G}^\sharp, \mathbf{G}^\sharp] = \mathrm{Res}_{F/\mathbb{Q}}\,\mathrm{Spin}(\mathbf{V}),$$

and the short exact sequence (4.1) induces a short exact sequence

$$1 \to \mathbf{Z}^{\mathbb{Q}} \to \mathbf{G}^{\sharp} \to \mathrm{Res}_{F/\mathbb{Q}}\mathbf{G} \to 1,$$

where

$$\mathbf{Z}^{\mathbb{Q}} = \{z \in \mathrm{Res}_{F(\daleth)/\mathbb{Q}}\mathrm{GL}_1 : \mathrm{Nm}_{F(\daleth)/F}(z) \in \mathbb{Q}^{\times}\}.$$

- In Case U, following [RSZ20], let $\mathrm{GU}^{\mathbb{Q}}(\mathbf{V})$ be the reductive group over $\mathbb{Q}$ defined by

  $$\mathrm{GU}^{\mathbb{Q}}(\mathbf{V}) = \{(g,\lambda) \in \mathrm{GL}(\mathbf{V}) \times \mathrm{GL}_1 : \langle gv, gw\rangle = \lambda \langle v, w\rangle\}.$$

  It is equipped with a similitude character

  $$\nu : \mathrm{GU}^{\mathbb{Q}}(\mathbf{V}) \to \mathrm{GL}_1, \qquad (g,\lambda) \mapsto \lambda.$$

  Note that $\mathrm{GU}^{\mathbb{Q}}(\mathbf{V})$ is a subgroup of the Weil restriction of the unitary similitude group $\mathrm{GU}(\mathbf{V})$. Define

  $$\mathbf{Z}^{\mathbb{Q}} = \{z \in \mathrm{Res}_{F_1/\mathbb{Q}}\mathrm{GL}_1 : \mathrm{Nm}_{F_1/F}(z) \in \mathbb{Q}^{\times}\}.$$

  It is naturally equipped with a map to GL_1. We then define a reductive group $\mathbf{G}^{\sharp}$ over $\mathbb{Q}$ by

  $$\mathbf{G}^{\sharp} = \mathrm{GU}^{\mathbb{Q}}(\mathbf{V}) \times_{\mathrm{GL}_1} \mathbf{Z}^{\mathbb{Q}}.$$

  It is isomorphic to $\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G} \times \mathbf{Z}^{\mathbb{Q}}$ via the isomorphism

  $$\mathbf{G}^{\sharp} \cong \mathrm{Res}_{F/\mathbb{Q}}\mathbf{G} \times \mathbf{Z}^{\mathbb{Q}} : (g,z) \mapsto (z^{-1}g, z).$$

4.2. **Endoscopic classification of automorphic representations.** Let $\mathbf{G}^*$ be as in §4.1 and $(\mathbf{G}, \boldsymbol{\varrho}, \boldsymbol{z})$ be a pure inner form of $\mathbf{G}^*$. To prepare for the Langlands–Kottwitz method in §4.5, we recall some results on endoscopic classifications of automorphic representations of orthogonal and unitary groups over a totally real field, following [Art13, KMSW14, Ish24, CZ25].

For each Archimedean place τ of F, fix a maximal compact subgroup $\mathscr{K}_\tau$ of $\mathbf{G}(F_\tau)$. For all but finitely many finite places v of F, the inner twist (ϱ_v, z_v) is trivial, and we fix a hyperspecial subgroup $\mathscr{K}_v^{\mathrm{hs}}$ of $\mathbf{G}(F_v)$ compatible with $\mathfrak{w}_v$, the localization at v of the Whittaker datum $\mathfrak{w}$. In Case O2, there exists a nontrivial outer automorphism ς of $\mathbf{G}^*$ which preserves $\mathfrak{w}$, thus also $\mathscr{K}_v^{\mathrm{hs}}$; in the other cases we take $\varsigma = 1$. Let $\mathcal{A}(\mathbf{G})$ denote the space of automorphic forms for $\mathbf{G}$ in the sense of [BJ79], with respect to the fixed compact subgroup $\mathscr{K}_\infty := \prod_{\tau\in\Sigma_F^\infty} \mathscr{K}_\tau$. We write $\mathcal{A}_2(\mathbf{G})$ for the space of square-integrable automorphic forms for $\mathbf{G}$. It is a module over the restricted tensor product

$$\tilde{\mathcal{H}}(\mathbf{G}(\mathbf{A}_F)) := \bigotimes\nolimits_v' \tilde{\mathcal{H}}(\mathbf{G}(F_v))$$

of the ς-invariants of local Hecke algebras with respect to the characteristic function of $\mathscr{K}_v^{\mathrm{hs}}$ (defined for all but finitely many finite places v of F).

The space $\mathcal{A}_2(\mathbf{G})$ can be decomposed into near equivalence classes of representations. Two irreducible representations $\pi = \otimes_v' \pi_v$ and $\pi' = \otimes_v' \pi_v'$ are called nearly equivalent if π_v and π_v' are isomorphic for all but finitely many places $v \in \Sigma_F$. The decomposition into near equivalence classes will be expressed in terms of elliptic global A-parameters. An elliptic global A-parameter $\boldsymbol{\psi} \in \Psi_{\mathrm{ell}}(\mathbf{G})$ is a formal finite sum of pairs

$$\boldsymbol{\psi} = \sum_i (\Pi_i, d_i),$$

where each Π_i is an irreducible cuspidal automorphic representation of $\mathrm{GL}_{n_i}(\mathbf{A}_{F_1})$ that is conjugate self-dual of sign $(-1)^{d_i-1}b(\mathbf{G})$ (defined similarly as in §2.2), such that

- $\sum_i n_i d_i = N(\mathbf{G})$.
- $(\Pi_i, d_i) \neq (\Pi_j, d_j)$ if $i \neq j$,

- In Case O2, we assume $\prod_i \omega_i^{d_i} = \chi_{F(\sqrt{\mathrm{disc}(\mathbf{G})})/F}$, where ω_i is the central character of Π_i, and $\chi_{F(\sqrt{\mathrm{disc}(\mathbf{G})})/F}$ is the quadratic character of $\mathbf{A}_F^\times/F^\times$ corresponding to the extension $F(\sqrt{\mathrm{disc}(\mathbf{G})})/F$ via global class field theory.

The parameter $\boldsymbol{\psi}$ is called *generic* (or *tempered*) if $d_i = 1$ for all i, in which case we abbreviate $(\Pi_i, 1)$ to Π_i.

Given an elliptic global A-parameter

$$\boldsymbol{\psi} = (\Pi_1, d_1) + \cdots + (\Pi_k, d_k)$$

for $\mathbf{G}$, define the formal extended component group

$$\mathfrak{S}^\sharp_{\boldsymbol{\psi}} := \bigoplus_i (\mathbb{Z}/2)e_i,$$

where e_i is a formal coordinate associated with the summand Π_i. In Case O2, there is a map

$$\det_{\boldsymbol{\psi}} : \mathfrak{S}^\sharp_{\boldsymbol{\psi}} \to \mathbb{Z}/2, \qquad \sum_{1\le i\le k} x_i e_i \mapsto \sum_{1\le i\le k} n_i d_i x_i,$$

and we define the formal component group $\mathfrak{S}_{\boldsymbol{\psi}} := \ker(\det_{\boldsymbol{\psi}})$. To unify notation, in Case O1 and Case U, we set $\mathfrak{S}_{\boldsymbol{\psi}} = \mathfrak{S}^\sharp_{\boldsymbol{\psi}}$. Moreover, we define the quotient group

$$\overline{\mathfrak{S}}_{\boldsymbol{\psi}} := \mathfrak{S}_{\boldsymbol{\psi}} / \langle e_1 + \cdots + e_k \rangle .$$

Finally, Arthur defines a canonical character $\varepsilon_{\boldsymbol{\psi}}$ of $\mathfrak{S}_{\boldsymbol{\psi}}$ as in [Art13, Equation (1.5.6)], which is trivial if $\boldsymbol{\psi}$ is generic.

For each $v \in \Sigma_F$, we can define the formal localization $\tilde{\boldsymbol{\psi}}_v := \sum_i (\phi_{i,v}, d_i)$: If $\Sigma_{F_1}(v)$ is a singleton, which we also denote by v, then $\phi_{i,v}$ is a n_i-dimensional representation of $W_{(F_1)_v} \times \mathrm{SL}_2(\mathbb{C})$ that corresponds to $\Pi_{i,v}$ via the local Langlands correspondence, which is conjugate self-dual of sign $(-1)^{d_i-1}b(\mathbf{G})$. We can associate to $\tilde{\boldsymbol{\psi}}_v$ a formal sum

$$\tilde{\phi}^{\mathrm{GL}}_{\tilde{\boldsymbol{\psi}}_v} := \sum_i \left(\left(\phi_{i,v} \otimes |-|_{(F_1)_v}^{\frac{d_i-1}{2}}\right) \oplus \left(\phi_{i,v} \otimes |-|_{(F_1)_v}^{\frac{d_i-3}{2}}\right) \oplus \cdots \oplus \left(\phi_{i,v} \otimes |-|_{(F_1)_v}^{\frac{1-d_i}{2}}\right) \right),$$

which may be regarded as an element of $\tilde{\Phi}(\mathbf{G}_v)$.

On the other hand, if $\#\Sigma_{F_1}(v) = 2$, then we are in Case U. If we write $\Sigma_{F_1}(v) = \{w, w^{\mathsf{c}}\}$, then $\Pi_{i,w} \cong \Pi^\vee_{i,w^{\mathsf{c}}}$ under the identifications $\mathrm{GL}_{n_i}(F_{1,w}) \cong \mathrm{GL}_{n_i}(F_{1,w^{\mathsf{c}}})$. We define $\phi_{i,v}$ to be the n_i-dimensional representation of $W_{F_v} \times \mathrm{SL}_2(\mathbb{C})$ that corresponds to $\Pi_{i,w}$ under the identification $w : F_{1,w} \xrightarrow{\sim} F_v$. This is independent of the choice of w after taking into account the corresponding change of the identification $\mathbf{G}_v \simeq \mathrm{GL}_{N(\mathbf{G}),F_v}$. We define $\tilde{\phi}^{\mathrm{GL}}_{\tilde{\boldsymbol{\psi}}_v}$ as before.

We then have the following theorem, usually called "Arthur's multiplicity formula":

Theorem 4.2.1 ([Art13, Mok15, KMSW14, Ish24, CZ25])**.** *If $\boldsymbol{\psi}$ is an elliptic global A-parameter for $\mathbf{G}$, we denote by $\mathcal{A}_{2,\boldsymbol{\psi}}(\mathbf{G})$ the direct sum of irreducible admissible representations π in $\mathcal{A}_2(\mathbf{G})$ such that $\tilde{\phi}^{\mathrm{GL}}_{\pi_v} \cong \tilde{\phi}^{\mathrm{GL}}_{\tilde{\boldsymbol{\psi}}_v}$ for all but finitely many places $v \in \Sigma_F$ (here if $\#\Sigma_{F_1}(v) = 2$ with $\Sigma_{F_1}(v) = \{w, w^{\mathsf{c}}\}$, then we write $\tilde{\phi}^{\mathrm{GL}}_{\pi_v}$ for the classical L-parameter of $\mathbf{G}_v \cong \mathrm{GL}_{N(\mathbf{G}),F_v}$ corresponding to π_v composed with the identification $w : F_{1,w} \xrightarrow{\sim} F_v$). Then there is a natural decomposition of $\tilde{\mathcal{H}}(\mathbf{G}(\mathbf{A}_F))$-modules*

$$\mathcal{A}_2(\mathbf{G}) = \bigoplus_{\boldsymbol{\psi}} \mathcal{A}_{2,\boldsymbol{\psi}}(\mathbf{G}),$$

where $\boldsymbol{\psi}$ runs through elliptic global A-parameters for $\mathbf{G}$. An irreducible subrepresentation π of $\mathcal{A}_{2,\boldsymbol{\psi}}(\mathbf{G})$ is said to have formal parameter $\boldsymbol{\psi}$.

Furthermore, for each generic elliptic global A-parameter $\boldsymbol{\psi} = \Pi_1 + \cdots + \Pi_k$ for $\mathbf{G}$, there exists a natural diagonal map

$$\Delta : \mathfrak{S}_{\boldsymbol{\psi}} \to \mathfrak{S}_{\boldsymbol{\psi},\mathbf{A}_F} := \prod_{v\in\Sigma_F} \mathfrak{S}_{\tilde{\phi}_{\tilde{\boldsymbol{\psi}}_v}},$$

and a decomposition of $\tilde{\mathcal{H}}(\mathbf{G}(\mathbf{A}_F))$-modules:

$$\mathcal{A}_{2,\boldsymbol{\psi}}(\mathbf{G}) \cong m_{\boldsymbol{\psi}} \bigoplus_{\substack{\eta\in\mathrm{Irr}(\mathfrak{S}_{\boldsymbol{\psi},\mathbf{A}_F})\\ \Delta^*(\eta)=\varepsilon_{\boldsymbol{\psi}}}} \tilde{\pi}_{\mathfrak{w},z}(\eta),$$

where each $\tilde{\pi}_{\mathfrak{w},z}(\eta) = \otimes_v \tilde{\pi}_{\mathfrak{w}_v,z_v}(\tilde{\boldsymbol{\psi}}_v, \eta_v)$ is the global restricted tensor product of local representations, cf. Theorem 2.3.1*. Moreover, the multiplicity $m_{\boldsymbol{\psi}}$ satisfies $m_{\boldsymbol{\psi}} = 1$ unless we are in case* O2 *and n_i is even for each i, in which case $m_{\boldsymbol{\psi}} = 2$.*

Proof. In Case O1, this is established in [Art13] when $\mathbf{G}$ is quasisplit, and established in [Ish24, Theorem 3.16, 3.17] when $\mathbf{G}$ is non-quasisplit. In Case O2, this is established in [Art13] when $\mathbf{G}$ is quasisplit, and in [CZ25, Theorem 2.1, 2.6] when $\mathbf{G}$ is non-quasisplit. In Case U, this is established in [Mok15] when $\mathbf{G}$ is quasisplit and established in [KMSW14, Theorem 1.7.1] when $\mathbf{G}$ is non-quasisplit. □

This theorem implies the following result on strong functorial transfer and strong multiplicity one for cuspidal automorphic representations of $\mathbf{G}(\mathbf{A}_F)$:

Corollary 4.2.2. *Let π be a cuspidal automorphic representation of $\mathbf{G}(\mathbf{A}_F)$ whose formal parameter $\boldsymbol{\psi}$ is generic. Then, for every finite place $v \in \Sigma_F^{\mathrm{fin}}$, one has*

$$\tilde{\phi}^{\mathrm{GL}}_{\pi_v} \cong \tilde{\phi}^{\mathrm{GL}}_{\tilde{\boldsymbol{\psi}}_v}.$$

We write $\pi^{\mathrm{GL}} := \boldsymbol{\psi}$ and call it the strong functorial transfer *of π.*

Moreover, if π' is another cuspidal automorphic representation of $\mathbf{G}(\mathbf{A}_F)$ or $\mathbf{G}^*(\mathbf{A}_F)$ with formal parameter π^{GL}, then

$$\tilde{\phi}_{\pi'_v} \cong \tilde{\phi}_{\pi_v}$$

for every finite place v of F, and π^{GL} is also the strong functorial transfer of π'.

4.3. **Controlled cuspidal automorphic representations.** Let $\mathbf{G}^*$ be as in §4.1. To specify the local conditions imposed on the automorphic representations under consideration, we will use the following notion of control tuples:

Definition 4.3.1. A *control tuple* for $\mathbf{G}^*$ is a tuple $\bigstar = (\Sigma^\circ, \Sigma^{\mathrm{St}}, \Sigma^{\mathrm{sc}}, \Sigma, \xi)$ where

- Σ^{St} and Σ^{sc} are disjoint nonempty finite sets of finite places of F.
- $\Sigma^\circ \subset \Sigma^{\mathrm{St}} \cup \Sigma^{\mathrm{sc}}$ and $\Sigma^{\mathrm{St}} \cup \Sigma^{\mathrm{sc}} \cup \Sigma_F^\infty \subset \Sigma$ are finite sets of places of F.
- $\xi = \otimes_{\tau\in\mathrm{Hom}(F,\mathbb{C})}\xi_\tau$ is an irreducible representation of $\left(\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G}^*\right) \otimes_{\mathbb{Q}} \mathbb{C}$ with regular algebraic highest weight.
- $\mathbf{G}^*_v$ is unramified for any finite place $v \in \Sigma_F^{\mathrm{fin}}$ not in Σ.

Definition 4.3.2. Let $\bigstar$ be a control tuple for $\mathbf{G}^*$. A pure inner twist $(\mathbf{G}, \boldsymbol{\varrho}, \boldsymbol{z})$ of $\mathbf{G}^*$ over F is called a $\bigstar$*-good pure inner form* of $\mathbf{G}^*$ if (ϱ_v, z_v) is trivial for each $v \in \Sigma_F^{\mathrm{fin}} \smallsetminus \Sigma^\circ$.

If $(\mathbf{G}, \boldsymbol{\varrho}, \boldsymbol{z})$ is a $\bigstar$-good pure inner form of $\mathbf{G}^*$, then for each $v \in \Sigma_F^{\mathrm{fin}} \smallsetminus \Sigma$, $\mathbf{G}_v$ has a reductive integral model $\mathcal{G}_v$ over $\mathcal{O}_{F_v}$ coming from the fixed reductive integral model $\mathcal{G}^*_v$ of $\mathbf{G}^*_v$ via ϱ_v. We also write $\mathcal{K}_v^{\mathrm{hs}}$ for the corresponding hyperspecial maximal compact subgroup and define the abstract Hecke algebra away from Σ:

$$\tilde{\mathbb{T}}^\Sigma := {\bigotimes}'_{v\in\Sigma_F^{\mathrm{fin}}\smallsetminus\Sigma} \tilde{\mathcal{H}}(\mathbf{G}(F_v), \mathcal{K}_v^{\mathrm{hs}}).$$

Definition 4.3.3. Let $\bigstar$ be a control tuple for $\mathbf{G}^*$ and let $(\mathbf{G}, \boldsymbol{\varrho}, \boldsymbol{z})$ be a $\bigstar$-good pure inner twist of $\mathbf{G}^*$. Suppose $\Sigma' \subset \Sigma^\circ$ is a subset. A compact open subgroup $\mathcal{K}^{\Sigma'} \le \mathbf{G}(\mathbf{A}_{F,f}^{\Sigma'})$ is called a $\bigstar$-*split subgroup* if it is of the form

$$\mathcal{K}^{\Sigma'} = \prod_{v \in \Sigma_F^{\mathrm{fin}} \smallsetminus \Sigma'} \mathcal{K}_v,$$

where $\mathcal{K}_v = \mathcal{K}_v^{\mathrm{hs}}$ for v not in Σ.

Definition 4.3.4. For a control tuple $\bigstar$ for $\mathbf{G}^*$ and a $\bigstar$-good pure inner twist $(\mathbf{G}, \boldsymbol{\varrho}, \boldsymbol{z})$ of $\mathbf{G}^*$, a $\bigstar$-*good automorphic representation* of $\mathbf{G}(\mathbf{A}_F)$ is a cuspidal automorphic representation $\pi = \otimes_v' \pi_v$ of $\mathbf{G}(\mathbf{A}_F)$ such that

- π_v is unramified for all $v \in \Sigma_F^{\mathrm{fin}} \smallsetminus \Sigma$;
- π_v is an unramified twist of the Steinberg representation for any $v \in \Sigma^{\mathrm{St}}$ (see Definition A.4.6);
- π_v has supercuspidal classical L-parameter for each $v \in \Sigma^{\mathrm{sc}}$. Moreover, π_v has simple classical L-parameter for some $v \in \Sigma^{\mathrm{sc}}$ (as defined in §2.2);
- π_∞ is cohomological for ξ, i.e.,

  $$\mathrm{H}^i(\mathrm{Lie}(\mathbf{G}(F \otimes \mathbb{R})), \mathcal{K}_\infty, \pi_\infty \otimes_{\mathbb{C}} \xi) \neq 0$$

  for some $i \in \mathbb{N}$.

We then have the following controlled strong transfer result:

Theorem 4.3.5. *For any control tuple $\bigstar$ for $\mathbf{G}^*$ and $\bigstar$-good pure inner twists $(\mathbf{G}, \boldsymbol{\varrho}, \boldsymbol{z}), (\mathbf{G}', \boldsymbol{\varrho}', \boldsymbol{z}')$ of $\mathbf{G}^*$, if π is a $\bigstar$-good automorphic representation of $\mathbf{G}$, then there exists a $\bigstar$-good automorphic representation τ of $\mathbf{G}'$ such that*

- *$\tau^\Sigma \cong \pi^\Sigma$ as $\tilde{\mathbb{T}}^\Sigma$-modules via the isomorphism*

  $$\boldsymbol{\varrho}' \circ \boldsymbol{\varrho}^{-1} : \mathbf{G}^\Sigma \xrightarrow{\sim} (\mathbf{G}')^\Sigma.$$

- *for any $v \in \Sigma_F^{\mathrm{fin}}$, τ_v has the same classical L-parameter as π_v.*

*Such a τ is called a $\bigstar$-*good transfer *of π to $\mathbf{G}'$.*

Proof. Let $\boldsymbol{\psi} = \pi^{\mathrm{GL}}$. By Definition 4.3.4, $\boldsymbol{\psi}$ is a simple generic elliptic global A-parameter. For each finite place v, let $\tilde{\phi}_v$ be the classical L-parameter of π_v.

If $v \notin \Sigma^\circ$, the two pure inner twists are locally identified, and we choose the member of $\tilde{\Pi}_{\tilde{\phi}_v}(\mathbf{G}'_v)$ corresponding to π_v. If $v \in \Sigma^\circ$, then $v \in \Sigma^{\mathrm{St}} \cup \Sigma^{\mathrm{sc}}$, so $\tilde{\phi}_v$ is discrete. The local packet parametrization in Theorem 2.3.1 therefore gives a nonempty packet $\tilde{\Pi}_{\tilde{\phi}_v}(\mathbf{G}'_v)$. At every Archimedean place v, choose a member of the cohomological discrete-series packet associated with ξ_v.

Let η_v denote the character of the local component group corresponding to the chosen local representation. Since $\boldsymbol{\psi}$ is simple, $\mathfrak{S}_{\boldsymbol{\psi}}$ has order 2 and is generated by its distinguished central element. The value of η_v on its localization is the local Kottwitz sign determined by (ϱ'_v, z'_v). The product formula for the global pure inner twist $(\mathbf{G}', \boldsymbol{\varrho}', \boldsymbol{z}')$ therefore gives

$$\Delta^*\big(\otimes_v \eta_v\big) = \mathbf{1} = \varepsilon_{\boldsymbol{\psi}}.$$

Arthur's multiplicity formula, Theorem 4.2.1, now produces a discrete automorphic representation

$$\tau = \otimes_v' \tau_v$$

of $\mathbf{G}'(\mathbf{A}_F)$ having the prescribed local members.

For every $v \in \Sigma^{\mathrm{sc}}$, the parameter $\tilde{\phi}_v$ is supercuspidal, and hence τ_v is supercuspidal. Since $\Sigma^{\mathrm{sc}} \neq \varnothing$, it follows that τ is cuspidal. The unramified and supercuspidal conditions are preserved because the local parameters are unchanged.

Finally, if $v \in \Sigma^{\mathrm{St}}$, then $\mathfrak{S}_{\tilde{\phi}_v}$ has order 2 and $z_{\tilde{\phi}_v}$ is nontrivial. Hence the relevant packet on $\mathbf{G}'_v$ is a singleton, whose unique member is an unramified twist of the Steinberg representation. Thus τ is $\bigstar$-good and has the asserted local properties. $\square$

We now recall the following result, due to the work of Clozel, Kottwitz, Harris–Taylor [HT01], Shin [Shi11], Chenevier–Harris [CH13] and others, which allows us to construct ℓ-adic representations attached to $\bigstar$-good automorphic representations:

Theorem 4.3.6 ([Clo90, HT01, TY07, Shi11, Car12, CH13, Car14])**.** *For any control tuple $\bigstar$ for $\mathbf{G}^*$ and $\bigstar$-good pure inner twist $(\mathbf{G}, \boldsymbol{\varrho}, \boldsymbol{z})$ of $\mathbf{G}^*$, if π is a $\bigstar$-good automorphic representation of $\mathbf{G}$, then π_v is tempered for every $v \in \Sigma_F^{\mathrm{fin}}$, and there exists a continuous irreducible representation, unique up to isomorphism,*

$$\rho_{\pi,\ell} : \mathrm{Gal}_{F_1} \to \mathrm{GL}_{N(\mathbf{G})}(\overline{\mathbb{Q}_\ell}),$$

such that

$$\mathrm{WD}\left(\rho_{\pi,\ell}|_{W_{F_{1,w}}}\right)^{F\text{-ss}} \cong \iota_\ell\left(\tilde{\phi}^{\mathrm{GL}}_{\pi_v} \otimes |-|_{F_{1,w}}^{\frac{1-N(\mathbf{G})}{2}}\right) \tag{4.2}$$

for any finite place w of F_1 with underlying finite place v of F. Here $\tilde{\phi}_{\pi_v}$ is the classical L-parameter of π_v. Note that, if $\#\Sigma_{F_1}(v) = 2$ with $\Sigma_{F_1}(v) = \{w, w^{\mathsf{c}}\}$ then we write $\tilde{\phi}^{\mathrm{GL}}_{\pi_v}$ for the classical L-parameter of $\mathbf{G}_v \cong \mathrm{GL}_{N(\mathbf{G}),F_v}$ corresponding to π_v composed with the identification $F_v \cong F_{1,w}$.

Moreover, if π' is another cuspidal automorphic representation of $\mathbf{G}(\mathbf{A}_F)$ or $\mathbf{G}^*(\mathbf{A}_F)$ such that π_v and π'_v have the same classical L-parameter $\tilde{\phi}_{\pi_v} = \tilde{\phi}_{\pi'_v}$ for all but finitely many finite places v of F, then Equation (4.2) *is satisfied for each $v \in \Sigma_F^{\mathrm{fin}}$ with π_v replaced by π'_v.*

Proof. Let π^{GL} be the strong functorial transfer of π to $\mathbf{G}^{\mathrm{GL}}$ (Corollary 4.2.2). Then π^{GL} is conjugate self-dual and cohomological with regular highest weight. We let $\rho_{\pi,\ell}$ be the Galois representation associated with π^{GL}. Such a $\rho_{\pi,\ell}$ is constructed in [CH13, Theorem 3.2.3] and the local-global compatibility is established in [Car12, Theorem 1.1] and [Car14, Theorem 1.1]. The temperedness is established for Archimedean places by Clozel [Clo90, Lemma 4.9] and established for finite places by [HT01, TY07, Shi11, Car12, Clo13, Car14]. The irreducibility follows from local-global compatibility and from the existence of a place $v \in \Sigma^{\mathrm{sc}}$ at which π_v has simple supercuspidal classical L-parameter.

The last assertion follows from Corollary 4.2.2. $\square$

4.4. **The Shimura data.** We first define the relevant Shimura varieties following [MP16] and [RSZ20].

Let F_1/F, $\daleth \in F_1^\times$ and $\mathbf{V}, \mathbf{G}, \mathbf{G}^*, \mathbf{G}^\sharp$ be as in §4.1.

Definition 4.4.1. A pure inner twist $(\mathbf{V}, \mathbf{G} = \mathrm{U}(\mathbf{V})^\circ)$ of $\mathbf{G}^*$ is called

- *standard definite* if $\mathbf{V} \otimes_{F,\tau} \mathbb{R}$ is positive definite for each $\tau : F \to \mathbb{R}$.
- *standard indefinite* if $\mathbf{V} \otimes_{F,\tau} \mathbb{R}$ has signature $(\dim \mathbf{V} - 2, 2)$ (resp. $(\dim \mathbf{V} - 1, 1)$) in Case O (resp. in Case U) for $\tau = \tau_0$ and positive definite (i.e., signature $(\dim \mathbf{V}, 0)$) for each $\tau \in \Sigma_F^\infty \smallsetminus \{\tau_0\}$.

Suppose $(\mathbf{V}, \mathbf{G} = \mathrm{U}(\mathbf{V})^\circ)$ is a standard indefinite pure inner twist of $\mathbf{G}^*$, and $\mathbf{G}^\sharp$ is the central extension of $\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G}$ defined in §4.1. We have the Shimura datum $(\mathbf{G}^\sharp, \mathbf{X}^\sharp)$, where $\mathbf{X}^\sharp$ is the conjugacy class of a Deligne homomorphism

$$h_0^\sharp : \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_1 \to \mathbf{G}^\sharp \otimes \mathbb{R} \tag{4.3}$$

defined as follows:

- In Case O, let
$$h_{0,\natural}:\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_1\to\Big(\mathrm{Res}_{F/\mathbb{Q}}\,\mathrm{GSpin}(\mathbf{V})\Big)\otimes\mathbb{R}\cong\prod_{\tau\in\mathrm{Hom}(F,\mathbb{R})}\mathrm{GSpin}(\mathbf{V}\otimes_{F,\tau}\mathbb{R})$$
be the homomorphism that is trivial on $\mathrm{GSpin}(\mathbf{V}\otimes_{F,\tau}\mathbb{R})$ for $\tau\neq\tau_0$, and on $\mathrm{GSpin}(\mathbf{V}\otimes_{F,\tau_0}\mathbb{R})$ it is induced by
$$h^{\sharp}_{0,\tau_0}:\mathbb{C}^{\times}\to\mathrm{GSpin}(\mathbf{V}\otimes_{F,\tau_0}\mathbb{R}):a+b\mathrm{i}\mapsto a+be_{\tau_0,1}e_{\tau_0,2}$$
where $e_{\tau_0,1},e_{\tau_0,2}$ are two orthogonal vectors in $\mathbf{V}\otimes_{F,\tau_0}\mathbb{R}$ such that $\|e_{\tau_0,1}\|=\|e_{\tau_0,2}\|=-1$. We also define
$$h_{0,\neg}:\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_1\to\Big(\mathrm{Res}_{F(\neg)/F}\mathrm{GL}_1\Big)\otimes\mathbb{R}\cong\prod_{\tau\in\mathrm{Hom}(F,\mathbb{R})}\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_1$$
to be the homomorphism that is trivial on the τ_0-factor and is the identity map on the other factors. We then define
$$h^{\sharp}_0=(h_{0,\natural},h_{0,\neg}):\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_1\to\mathbf{G}_{\neg}\otimes\mathbb{R},$$
which factors through $\mathbf{G}^{\sharp}\otimes\mathbb{R}\subset\mathbf{G}_{\neg}\otimes\mathbb{R}$.
- In Case U, for each $\tau\in\mathrm{Hom}(F,\mathbb{C})$, we fix a $\mathbb{C}$-basis $\mathbf{v}=(v_1,\dots,v_{n(G)})^{\top}$ of $\mathbf{V}\otimes_{F_1,\tau}\mathbb{C}$ such that
$$J_\tau:=(\langle v_i,v_j\rangle_{i,j})=\begin{cases}\mathrm{diag}(\underbrace{1,\dots,1}_{(n(G)-1)\text{-many}},-1) & \text{if }\tau=\tau_0\\ \mathbf{1} & \text{if }\tau\neq\tau_0\end{cases}.$$
Let
$$h_{0,\natural}:\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_1\to\mathrm{GU}^{\mathbb{Q}}(\mathbf{V})\otimes\mathbb{R}\cong\prod_{\tau\in\mathrm{Hom}(F,\mathbb{R})}\mathrm{GU}(\mathbf{V}\otimes_{F_1,\tau}\mathbb{C})$$
be the homomorphism such that on each τ-factor it is given by
$$a+b\mathrm{i}\mapsto a+b\mathrm{i}J_\tau.$$
Let
$$h_{0,\neg}:\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_1\to\mathbf{Z}^{\mathbb{Q}}\otimes\mathbb{R}\subset\prod_{\tau\in\mathrm{Hom}(F,\mathbb{R})}\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_1$$
be the diagonal embedding. We then define
$$h^{\sharp}_0=(h_{0,\natural},h_{0,\neg}):\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_1\to\mathbf{G}^{\sharp}\otimes\mathbb{R}.$$

It is routine to check that $(\mathbf{G}^{\sharp},\mathbf{X}^{\sharp})$ is a Shimura datum.

We define
$$h_0:\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_1\to\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G}$$
to be the composition of $h^{\sharp}_0$ with the central extension $\mathbf{G}^{\sharp}\to\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G}$, and let $\mathbf{X}:=\{h_0\}$ be the $G(F\otimes\mathbb{R})$-conjugacy class of h_0. Then $(\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G},\mathbf{X})$ is a Shimura datum. Its Hodge cocharacter is
$$\mu:\mathrm{GL}_{1,\mathbb{C}}\xrightarrow{z\mapsto(z,1)}\Big(\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathrm{GL}_1\Big)_{\mathbb{C}}\xrightarrow{(h_0)_{\mathbb{C}}}\Big(\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G}\Big)_{\mathbb{C}}.$$
Its reflex field E is F_1, except in Case U with $n(\mathbf{G})=2$, where $E=F$. We use the embedding $E\hookrightarrow\mathbb{C}$ induced by τ_0.

For any rational prime p and any fixed isomorphism $\iota_p:\mathbb{C}\xrightarrow{\sim}\overline{\mathbb{Q}_p}$, the induced cocharacter of
$$(\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G})_{\overline{\mathbb{Q}_p}}\cong\prod_{v:F\hookrightarrow\overline{\mathbb{Q}_p}}\mathbf{G}\otimes_{F,v}\overline{\mathbb{Q}_p}.$$

is conjugate to the inverse of the cocharacter μ_1 defined in (2.5) on the factor corresponding to $\iota_p \circ \tau_0|_F$, and is trivial on all other factors.

We define $\mathcal{K}_\infty$ and $\mathcal{K}_\infty^\sharp$ to be the centralizer of h_0 and $h_0^\sharp$, respectively. Then

$$\mathbf{X} \cong \mathbf{G}(F \otimes \mathbb{R})/\mathcal{K}_\infty, \qquad \mathbf{X}^\sharp \cong \mathbf{G}^\sharp(\mathbb{R})/\mathcal{K}_\infty^\sharp.$$

By Deligne's theory, there is a projective system

$$\{\mathrm{Sh}_{\mathcal{K}}(\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G}, \mathbf{X})\}$$

of Shimura varieties defined over E indexed by neat open compact subgroups $\mathcal{K} \leq \mathbf{G}(\mathbf{A}_{F,f})$ (as defined in [Pin90, §0.6]), with complex uniformization

$$\mathrm{Sh}_{\mathcal{K}}(\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G}, \mathbf{X}) \otimes_{E,\tau_0} \mathbb{C} \cong \mathbf{G}(F)\backslash \left(\mathbf{X} \times \mathbf{G}(\mathbf{A}_{F,f})/\mathcal{K}\right).$$

Its complex dimension is $\dim_{\mathbb{C}}(\mathbf{X})$. When $E = F$, we tacitly replace this system by its base change to F_1 and retain the same notation.

Similarly, there is a projective system

$$\{\mathrm{Sh}_{\mathcal{K}^\sharp}(\mathbf{G}^\sharp, \mathbf{X}^\sharp)\}_{\mathcal{K}^\sharp}$$

of Shimura varieties over the reflex field $E^\sharp$ indexed by neat open compact subgroups $\mathcal{K}^\sharp \leq \mathbf{G}^\sharp(\mathbf{A}_f)$.[9] Each member has complex dimension $\dim_{\mathbb{C}}(\mathbf{X})$. If $\mathcal{K}$ is the image of $\mathcal{K}^\sharp$ in $\mathbf{G}(\mathbf{A}_{F,f})$, then there is a morphism

$$\mathrm{Sh}_{\mathcal{K}^\sharp}(\mathbf{G}^\sharp, \mathbf{X}^\sharp) \longrightarrow \mathrm{Sh}_{\mathcal{K}}(\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G}, \mathbf{X})_{E^\sharp} \tag{4.4}$$

functorial in $\mathcal{K}^\sharp$.

Finally, we check that $(\mathbf{G}^\sharp, \mathbf{X}^\sharp)$ is a Shimura datum of Hodge type and thus $(\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G}, \mathbf{X})$ is a Shimura datum of Abelian type. In Case U, this follows from [RSZ20, §3.2], and $(\mathbf{G}^\sharp, \mathbf{X}^\sharp)$ is of PEL type. In Case O, let $c_\urcorner$ be the nontrivial element of $\mathrm{Gal}(F(\urcorner)/F)$. Let $\mathbf{H} = \mathrm{Cl}(\mathbf{V})$, viewed as an F-representation of $\mathrm{GSpin}(\mathbf{V})$ via left multiplication, and set

$$\mathbf{H}_\urcorner = \mathbf{H} \otimes_F F(\urcorner).$$

We define an F-linear anti-involution $\dagger$ on $\mathbf{H}_\urcorner$ by

$$(x \otimes a)^\dagger = x^* \otimes a^{c_\urcorner},$$

where $*$ is the main anti-involution on $\mathbf{H}$; see [MP16, §1.1]. The use of the conjugation $c_\urcorner$ is essential in order to obtain the norm character in the similitude factor below.

Let $\beta_\urcorner \in \mathbf{H}_\urcorner^\times$ satisfy

$$\beta_\urcorner^\dagger = -\beta_\urcorner.$$

We define an F-valued pairing on $\mathbf{H}_\urcorner$, viewed as an F-vector space, by

$$\psi_{\beta_\urcorner,\urcorner}(x, y) = \mathrm{tr}_{F(\urcorner)/F}\left(\mathrm{trrd}_{\mathbf{H}_\urcorner/F(\urcorner)}\left(x\beta_\urcorner y^\dagger\right)\right), \qquad x, y \in \mathbf{H}_\urcorner.$$

This pairing is nondegenerate and alternating. Indeed, using

$$\mathrm{trrd}_{\mathbf{H}_\urcorner/F(\urcorner)}(z^\dagger) = \mathrm{trrd}_{\mathbf{H}_\urcorner/F(\urcorner)}(z)^{c_\urcorner},$$

the invariance of $\mathrm{tr}_{F(\urcorner)/F}$ under $c_\urcorner$, and the identity $\beta_\urcorner^\dagger = -\beta_\urcorner$, we have

$$\begin{aligned}
\psi_{\beta_\urcorner,\urcorner}(y, x) &= \mathrm{tr}_{F(\urcorner)/F}\left(\mathrm{trrd}_{\mathbf{H}_\urcorner/F(\urcorner)}\left(y\beta_\urcorner x^\dagger\right)\right) \\
&= \mathrm{tr}_{F(\urcorner)/F}\left(\mathrm{trrd}_{\mathbf{H}_\urcorner/F(\urcorner)}\left((y\beta_\urcorner x^\dagger)^\dagger\right)\right) \\
&= \mathrm{tr}_{F(\urcorner)/F}\left(\mathrm{trrd}_{\mathbf{H}_\urcorner/F(\urcorner)}\left(x\beta_\urcorner^\dagger y^\dagger\right)\right) \\
&= -\psi_{\beta_\urcorner,\urcorner}(x, y).
\end{aligned}$$

[9]Note that the reflex field $E^\sharp$ may be bigger than F_1.

For $(g,t) \in \mathrm{GSpin}(\mathbf{V}) \times F(\daleth)^\times$, let (g,t) act on $\mathbf{H}_\daleth$ by

$$x \longmapsto gxt.$$

Then

$$\begin{aligned}
\psi_{\beta_\daleth,\daleth}(gxt, gyt) &= \mathrm{tr}_{F(\daleth)/F}\left(\mathrm{trrd}_{\mathbf{H}_\daleth/F(\daleth)}\left(gxt\beta_\daleth(gyt)^\dagger\right)\right) \\
&= \mathrm{tr}_{F(\daleth)/F}\left(\mathrm{trrd}_{\mathbf{H}_\daleth/F(\daleth)}\left(gx\, t\beta_\daleth t^{c_\daleth} y^\dagger g^*\right)\right) \\
&= \nu(g)\,\mathrm{Nm}_{F(\daleth)/F}(t)\,\mathrm{tr}_{F(\daleth)/F}\left(\mathrm{trrd}_{\mathbf{H}_\daleth/F(\daleth)}\left(x\beta_\daleth y^\dagger\right)\right) \\
&= \nu(g)\,\mathrm{Nm}_{F(\daleth)/F}(t)\,\psi_{\beta_\daleth,\daleth}(x,y),
\end{aligned}$$

where we used $g^*g = \nu(g)$ and the cyclicity of the reduced trace. The anti-diagonally embedded $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_1$ acts trivially on $\mathbf{H}_\daleth$, and the above action therefore descends to an embedding

$$\begin{aligned}
a : \mathbf{G}_\daleth = \left((\mathrm{Res}_{F/\mathbb{Q}}\,\mathrm{GSpin}(\mathbf{V})) \times \mathrm{Res}_{F(\daleth)/\mathbb{Q}}\mathrm{GL}_1\right)/\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_1 &\hookrightarrow \mathrm{Res}_{F/\mathbb{Q}}\,\mathrm{GSp}_F\left(\mathbf{H}_\daleth, \psi_{\beta_\daleth,\daleth}\right), \\
(g,t) &\longmapsto (x \mapsto gxt).
\end{aligned}$$

Moreover, if ν_{GSp} denotes the similitude character on $\mathrm{GSp}_F\left(\mathbf{H}_\daleth, \psi_{\beta_\daleth,\daleth}\right)$, then

$$\mathrm{Res}_{F/\mathbb{Q}}(\nu_{\mathrm{GSp}}) \circ a = \nu^\sharp.$$

Now define the $\mathbb{Q}$-valued symplectic form

$$\psi^{\mathbb{Q}}_{\beta_\daleth,\daleth} = \mathrm{tr}_{F/\mathbb{Q}} \circ \psi_{\beta_\daleth,\daleth}.$$

By the definition of $\mathbf{G}^\sharp$, the restriction of $\nu^\sharp$ to $\mathbf{G}^\sharp$ takes values in the diagonal subtorus

$$\mathrm{GL}_1 \subset \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_1.$$

Consequently, after viewing $\mathbf{H}_\daleth$ as a $\mathbb{Q}$-vector space, the restriction of a gives an embedding of reductive groups over $\mathbb{Q}$

$$\rho : \mathbf{G}^\sharp \hookrightarrow \mathrm{GSp}\left(\mathbf{H}_\daleth, \psi^{\mathbb{Q}}_{\beta_\daleth,\daleth}\right),$$

where the similitude factor is the above diagonal $\mathbb{Q}$-valued character.

It remains to choose $\beta_\daleth$ so that the above symplectic representation is a morphism of Shimura data. Equivalently, for every real embedding $\tau : F \hookrightarrow \mathbb{R}$, if

$$J_\tau = \rho(h_0^\sharp(\mathrm{i}))$$

is the induced complex structure on $\mathbf{H}_\daleth \otimes_{F,\tau} \mathbb{R}$, then the symmetric form

$$(x,y) \longmapsto \psi_{\beta_\daleth,\daleth,\tau}(x, J_\tau y)$$

must be definite.

Such a choice of $\beta_\daleth$ exists by the standard Kuga–Satake linear algebra. At the distinguished real place τ_0, where $\mathbf{V}_{\tau_0}$ has signature $(\dim \mathbf{V} - 2, 2)$, this is precisely the usual choice of a skew Clifford element giving a polarization of the Clifford representation; see [MP16, 3.5]. At the remaining real places $\tau \neq \tau_0$, the space $\mathbf{V}_\tau$ is positive definite and the complex structure comes only from the $F(\daleth)^\times$-factor, so a suitable real multiple of $1 \otimes \daleth$ gives the required definite form. These local positivity conditions define nonempty open subsets of

$$\{b \in \mathbf{H}_\daleth \otimes_{F,\tau} \mathbb{R} : b^\dagger = -b\}.$$

Together with the open condition of invertibility, weak approximation gives a global element

$$\beta_\daleth \in \mathbf{H}_\daleth^\times, \qquad \beta_\daleth^\dagger = -\beta_\daleth,$$

satisfying the required positivity at all Archimedean places.

For such a choice of $\beta_\neg$, the above embedding sends $h_0^\sharp$ to the Siegel double space attached to the symplectic space

$$\left(\mathbf{H}_\neg, \psi^{\mathbb{Q}}_{\beta_\neg,\neg}\right).$$

Hence it induces an embedding of Shimura data

$$(\mathbf{G}^\sharp, \mathbf{X}^\sharp) \hookrightarrow \left(\mathrm{GSp}\left(\mathbf{H}_\neg, \psi^{\mathbb{Q}}_{\beta_\neg,\neg}\right), \mathcal{X}\right),$$

where $\mathcal{X}$ is the union of Siegel upper half-spaces attached to $\left(\mathbf{H}_\neg, \psi^{\mathbb{Q}}_{\beta_\neg,\neg}\right)$. Thus $(\mathbf{G}^\sharp, \mathbf{X}^\sharp)$ is of Hodge type. Since $\mathbf{G}^\sharp$ is a central extension of $\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G}$, the Shimura datum $(\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G}, \mathbf{X})$ is of Abelian type.

4.5. **Langlands–Kottwitz method.** In this subsection, we apply the Langlands–Kottwitz method to relate the action of Frobenius elements at primes of good reduction to the Hecke action on the compactly supported cohomology of orthogonal or unitary Shimura varieties. We adopt the notation from §4.1, and assume:

Notation 4.5.1.

- F has a finite place $\mathfrak{q}$ whose residue characteristic is odd and which is inert in F_1 in Case U.
- $\bigstar$ is a control tuple (see Definition 4.3.1) such that $\mathfrak{q} \in \Sigma^{\mathrm{St}}$ and Σ is of the form $\Sigma = \Sigma_F(S_{\mathrm{bad}})$, where S_{bad} is a finite set of rational primes containing 2 and all rational primes ramified in F.
- $(\mathbf{G}, \boldsymbol{\varrho}, \boldsymbol{z})$ is a $\bigstar$-good pure inner form of $\mathbf{G}^*$.
- π is a $\bigstar$-good automorphic representation of $\mathbf{G}$.
- $\mathcal{K} \leq \mathbf{G}(\mathbf{A}_{F,f})$ is a $\bigstar$-split compact open subgroup with $\pi^{\mathcal{K}} \neq 0$.
- ℓ is a rational prime together with an isomorphism $\iota_\ell : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}_\ell}$.
- $(\mathfrak{X}, \chi)$ is a central character datum for $\mathbf{G}^*$ (see Definition A.4.2) with

$$\mathfrak{X} = (\boldsymbol{\varrho}^{-1}(\mathcal{K}) \cap Z_{\mathbf{G}^*}(\mathbf{A}_{F,f})) \times Z_{\mathbf{G}^*}(F \otimes \mathbb{R}),$$

  and χ is the inverse of the central character of ξ (extended from $Z_{\mathbf{G}^*}(F \otimes \mathbb{R})$ to $\mathfrak{X}$ trivially on $\boldsymbol{\varrho}^{-1}(\mathcal{K}) \cap Z_{\mathbf{G}^*}(\mathbf{A}_{F,f})$).

Definition 4.5.2. Suppose $(\mathbb{G}, \mathbb{X})$ is any Shimura datum with reflex field $E \subset \mathbb{C}$ with associated projective system of Shimura varieties $\{\mathrm{Sh}_{\mathcal{K}}(\mathbb{G}, \mathbb{X})\}_{\mathcal{K}}$ defined over E, indexed by the set of neat compact open subgroups $\mathcal{K} \leq \mathbb{G}(\mathbf{A}_f)$. Let Z_{a} be the maximal anisotropic $\mathbb{Q}$-subtorus of $Z(\mathbb{G})$, and let Z_{ac} be the smallest $\mathbb{Q}$-subgroup of Z_{a} whose base change to $\mathbb{R}$ contains the maximal $\mathbb{R}$-split subtorus of Z_{a}; see [KSZ21, Definition 1.5.4]. For each irreducible algebraic representation ξ of $\mathbb{G}_{\mathbb{C}}$ that is trivial on Z_{ac}, there exists a compatible system of lisse $\overline{\mathbb{Q}_\ell}$-local systems $\mathscr{L}_{\iota_\ell\xi}$ on this projective system of Shimura varieties associated with ξ; see [KSZ21, 1.5.8]. For each $i \in \mathbb{N}$, we define

$$\mathrm{H}^i_{\text{ét},\mathrm{c}}(\mathrm{Sh}, \mathscr{L}_{\iota_\ell\xi}) := \varinjlim_{\mathcal{K}\to\mathbf{1}} \mathrm{H}^i_{\text{ét},\mathrm{c}}\left(\mathrm{Sh}_{\mathcal{K}}(\mathbb{G}, \mathbb{X})_{\overline{E}}, \mathscr{L}_{\iota_\ell\xi}\right).$$

This is a $\mathbb{G}(\mathbf{A}_f) \times \mathrm{Gal}_E$-module with admissible $\mathbb{G}(\mathbf{A}_f)$-action and continuous Gal_E-action.

For each admissible representation Π of $\mathbb{G}(\mathbf{A})$, we define

$$\mathrm{H}^i_{\text{ét},\mathrm{c}}(\mathrm{Sh}, \mathscr{L}_{\iota_\ell\xi})[\Pi^\infty] := \mathrm{Hom}_{\mathbb{G}(\mathbf{A}_f)}\left(\iota_\ell\Pi^\infty, \mathrm{H}^i_{\text{ét},\mathrm{c}}(\mathrm{Sh}, \mathscr{L}_{\iota_\ell\xi})\right). \tag{4.5}$$

This is a finite dimensional representation of Gal_E, unramified at all but finitely many places. We set

$$\mathrm{H}^i_{\text{ét},\mathrm{c}}(\mathrm{Sh}, \mathscr{L}_{\iota_\ell\xi})^{\mathrm{ss}}[\Pi^\infty] := \left(\mathrm{H}^i_{\text{ét},\mathrm{c}}(\mathrm{Sh}, \mathscr{L}_{\iota_\ell\xi})[\Pi^\infty]\right)^{\mathrm{ss}}$$

for its semisimplification.

Similarly, if $\mathbb{G}'$ is any reductive group over $\mathbb{Q}$ such that $\mathbb{G}'(\mathbb{R})$ is compact, then for each irreducible algebraic representation (ξ, V_ξ) of $\mathbb{G}'_{\mathbb{C}}$, we define the injective system of *algebraic automorphic forms* valued in $\iota_\ell V_\xi$ as

$$\{\mathcal{A}\left(\mathbb{G}'(\mathbb{Q})\backslash\mathbb{G}'(\mathbf{A}_f)/\mathscr{K}, \mathscr{L}_{\iota_\ell\xi}\right)\}_{\mathscr{K}}$$

indexed by compact open subgroups $\mathscr{K} \leq \mathbb{G}'(\mathbf{A}_f)$, where $\mathcal{A}\left(\mathbb{G}'(\mathbb{Q})\backslash\mathbb{G}'(\mathbf{A}_f)/\mathscr{K}, \mathscr{L}_{\iota_\ell\xi}\right)$ consists of maps $\phi : \mathbb{G}'(\mathbf{A}_f) \to \iota_\ell V_\xi$ such that $\phi(gk) = \phi(g)$ and $\phi(\gamma g) = \gamma.\phi(g)$ for any $g \in \mathbb{G}'(\mathbf{A}_f), \gamma \in \mathbb{G}'(\mathbb{Q})$ and $k \in \mathscr{K}$. For each compact open subgroup $\mathscr{K} \leq \mathbb{G}'(\mathbf{A}_f^p)$, we write

$$\mathcal{A}\left(\mathbb{G}'(\mathbb{Q})\backslash\mathbb{G}'(\mathbf{A}_f)/\mathscr{K}^p, \mathscr{L}_{\iota_\ell\xi}\right) := \varinjlim_{\mathscr{K}_p} \mathcal{A}\left(\mathbb{G}'(\mathbb{Q})\backslash\mathbb{G}'(\mathbf{A}_f)/\mathscr{K}_p\mathscr{K}^p, \mathscr{L}_{\iota_\ell\xi}\right)$$

where $\mathscr{K}_p$ runs through compact open subgroups of $\mathbb{G}'(\mathbb{Q}_p)$.

The following theorem describes the Galois action on the cohomology of orthogonal or unitary Shimura varieties, which is the main result of [KSZ21] (cf. [KS23, Theorem 7.3]).

Theorem 4.5.3. *Suppose $\mathscr{K}_p \leq \mathbf{G}(F \otimes \mathbb{Q}_p)$ is hyperspecial for some $p \geq 3$, and $\iota_p : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}_p}$ is an isomorphism such that $\iota_p \circ \tau_0 : F_1 \to \overline{\mathbb{Q}_p}$ induces a finite place $\mathfrak{p} \in \Sigma_{F_1}(\{p\})$. Define a test function $f^\infty = f^{\infty,p} f_p \in \mathcal{H}\left(\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G}(\mathbf{A}_f), \mathscr{K}\right)$ with $f_p = \mathbf{1}_{\mathscr{K}_p}$. Then there exists $j_0 \in \mathbb{Z}_+$ such that*

$$\sum_{i=0}^{2\dim_{\mathbb{C}}(\mathbf{X})} (-1)^i \iota_\ell^{-1} \mathrm{Tr}\left(\iota_\ell f^\infty \sigma_{\mathfrak{p}}^j | \mathrm{H}^i_{\text{ét,c}}(\mathrm{Sh}, \mathscr{L}_{\iota_\ell\xi})\right) = \sum_{\mathfrak{e} \in \mathcal{E}_{\text{ell}}(\mathbf{G})} \iota(\mathfrak{e}) \mathrm{ST}^{\mathbf{G}^{\mathfrak{e}}}_{\text{ell},\chi}(h^{\mathbf{G}^{\mathfrak{e}}}_{\xi,j})$$

for all positive integers $j \geq j_0$. Here $\mathrm{ST}^{\mathbf{G}^{\mathfrak{e}}}_{\text{ell},\chi}$ is the elliptic stable distribution associated with $\mathfrak{e}$ (see Definition A.4.9*), $\iota(\mathfrak{e}) \in \mathbb{Q}$ is the global coefficient introduced by Kottwitz and Shelstad (cf.* [Art13, Equation (3.2.4)]*), and $h^{\mathbf{G}^{\mathfrak{e}}}_{\xi,j} = h^{\mathbf{G}^{\mathfrak{e}},p\infty} h^{\mathbf{G}^{\mathfrak{e}}}_{p,j} h^{\mathbf{G}^{\mathfrak{e}}}_{\infty,\xi} \in \mathcal{H}(\mathbf{G}^{\mathfrak{e}}(\mathbf{A}_F), \chi^{-1})$ are defined in* [Kot90, KSZ21]. *In particular, $\iota(\mathfrak{e}) = 1$ if $\mathbf{G}^{\mathfrak{e}} = \mathbf{G}^*$, and*

- *$h^{\mathbf{G}^*,p\infty}$ is an endoscopic transfer of $f^{p\infty}$, (Note that such a transfer exists in the fixed-central character setting, by first lifting f along the averaging map $\mathcal{H}(\mathbf{G}(\mathbf{A}_F)) \to \mathcal{H}(\mathbf{G}(\mathbf{A}_F), \chi^{-1})$, and then taking the transfer to $\mathcal{H}(\mathbf{G}^*(\mathbf{A}_F))$, and finally taking the image along the averaging map $\mathcal{H}(\mathbf{G}^*(\mathbf{A}_F)) \to \mathcal{H}(\mathbf{G}^*(\mathbf{A}_F), \chi^{-1})$),*
- *$h^{\mathbf{G}^*}_{p,j}$ is the base change transfer of*

$$\phi_j := \mathbf{1}_{\mathcal{G}_p\left(\mathbb{Z}_{\|\mathfrak{p}\|^j}\right)\mu(p^{-1})\mathcal{G}_p\left(\mathbb{Z}_{\|\mathfrak{p}\|^j}\right)} \in \mathcal{H}\left(\mathbf{G}(F \otimes \mathbb{Z}_{\|\mathfrak{p}\|^j}), \mathcal{G}_p\left(\mathbb{Z}_{\|\mathfrak{p}\|^j}\right)\right)$$

  (where we write $\mathcal{G}_p$ for $\prod_{v \in \Sigma_F(\{p\})} \mathrm{Res}_{\mathcal{O}_{F_v}/\mathbb{Z}_p}\mathcal{G}_v$ and write $\mathbb{Z}_{\|\mathfrak{p}\|^j}$ for the integer ring of the unramified extension $\mathbb{Q}_{\|\mathfrak{p}\|^j}$ of $\mathbb{Q}_p$ of degree $j \cdot \log_p(\|\mathfrak{p}\|)$) down to $\mathcal{H}\left(\mathbf{G}(F \otimes \mathbb{Q}_p), \mathcal{G}_p(\mathbb{Z}_p)\right)$,
- *$h^{\mathbf{G}^*}_{\infty,\xi} := \#\Pi_\xi(\mathbf{G}^*(F \otimes \mathbb{R}))^{-1} \sum_{\tau_\infty \in \Pi_\xi(\mathbf{G}^*(F\otimes\mathbb{R}))} f_{\tau_\infty}$, that is, the average of the pseudo-coefficients for the discrete series L-packet of $\mathbf{G}^*(F \otimes \mathbb{R})$ associated with ξ.*

Definition 4.5.4. Let $[\pi]$ be the set of isomorphism classes of $\bigstar$-good automorphic representations $\dot{\pi}$ of $\mathbf{G}(\mathbf{A}_F)$ such that $\tilde{\dot{\pi}}_v \cong \tilde{\pi}_v$ for each $v \in \Sigma_F^{\text{fin}}$. Consider two automorphic representations $\dot{\pi}_1, \dot{\pi}_2 \in [\pi]$ equivalent and write $\dot{\pi}_1 \sim \dot{\pi}_2$ if $\dot{\pi}_1^\infty \cong \dot{\pi}_2^\infty$. In particular, $[\pi]/\sim$ is a singleton in Case U and Case O1.

Choose one representative from each equivalence class in $[\pi]/\sim$. We define the virtual Galois representation

$$\rho^\pi_{\mathrm{Sh}} := (-1)^{\dim_{\mathbb{C}}(\mathbf{X})} \sum_{\dot{\pi} \in [\pi]/\sim} \sum_{i=0}^{2\dim_{\mathbb{C}}(\mathbf{X})} (-1)^i \mathrm{H}^i_{\text{ét,c}}(\mathrm{Sh}, \mathscr{L}_{\iota_\ell\xi})^{\mathrm{ss}}[\dot{\pi}^\infty] \in \mathrm{K}_0\left(\overline{\mathbb{Q}_\ell}[\mathrm{Gal}_{F_1}]\right),$$

where $\mathrm{K}_0\left(\overline{\mathbb{Q}_\ell}[\mathrm{Gal}_{F_1}]\right)$ is the Grothendieck group of finite dimensional continuous representations of Gal_{F_1} with $\overline{\mathbb{Q}_\ell}$-coefficients unramified at all but finitely many places.

We will also need the following cohomology spaces to deal with non-compact Shimura varieties: Let

$$\mathrm{H}^i_{(2)}(\mathrm{Sh}, \mathscr{L}_\xi) := \varinjlim_{\mathcal{K}\to 1} \mathrm{H}^i_{(2)}(\mathrm{Sh}_{\mathcal{K}}(\mathbf{G},\mathbf{X}), \mathscr{L}_\xi)$$

be the L^2-cohomology of $\mathrm{Sh}(\mathbf{G},\mathbf{X}) \times_{F_1,\tau_0} \mathbb{C}$ as defined in [Fal83, §6], and let

$$\mathrm{IH}^*(\mathrm{Sh}, \mathscr{L}_{\iota_\ell\xi}) := \varinjlim_{\mathcal{K}\to 1} \mathrm{IH}^*(\mathrm{Sh}_{\mathcal{K}}(\mathbf{G},\mathbf{X}), \mathscr{L}_{\iota_\ell\xi})$$

be the ℓ-adic intersection cohomology of $\mathrm{Sh}(\mathbf{G},\mathbf{X})$. These two cohomologies are equipped with admissible $\mathbf{G}(\mathbf{A}_{F,f})$-actions defined by Hecke correspondences. There are natural $\mathbf{G}(\mathbf{A}_{F,f})$-equivariant maps

$$\begin{gathered}\mathrm{H}^i_{\text{ét,c}}(\mathrm{Sh}, \mathscr{L}_{\iota_\ell\xi}) \to \iota_\ell \mathrm{H}^i_{(2)}(\mathrm{Sh}, \mathscr{L}_\xi) \to \mathrm{H}^i_{\text{ét}}(\mathrm{Sh}, \mathscr{L}_{\iota_\ell\xi}), \\ \mathrm{H}^i_{\text{ét,c}}(\mathrm{Sh}, \mathscr{L}_{\iota_\ell\xi}) \to \mathrm{IH}^i(\mathrm{Sh}, \mathscr{L}_{\iota_\ell\xi}),\end{gathered} \tag{4.6}$$

and it follows from Zucker's conjecture [Loo88, LR91, SS90] that there is a $\mathbf{G}(\mathbf{A}_{F,f})$-equivariant commutative diagram

$$\begin{tikzcd} \mathrm{H}^i_{\text{ét,c}}(\mathrm{Sh}, \mathscr{L}_{\iota_\ell\xi}) \arrow[r] \arrow[dr] & \mathrm{IH}^i(\mathrm{Sh}, \mathscr{L}_{\iota_\ell\xi}) \arrow[d, "\cong"] \\ & \iota_\ell \mathrm{H}^i_{(2)}(\mathrm{Sh}, \mathscr{L}_\xi) \end{tikzcd} . \tag{4.7}$$

Lemma 4.5.5. *The maps in* (4.6) *induce isomorphisms*

$$\mathrm{H}^i_{\text{ét,c}}(\mathrm{Sh}, \mathscr{L}_{\iota_\ell\xi})[\pi^\infty] \cong \iota_\ell \mathrm{H}^i_{(2)}(\mathrm{Sh}, \mathscr{L}_\xi)[\pi^\infty] \cong \mathrm{H}^i_{\text{ét}}(\mathrm{Sh}, \mathscr{L}_{\iota_\ell\xi})[\pi^\infty].$$

Moreover, $\dim \mathrm{H}^i_{\text{ét,c}}(\mathrm{Sh}, \mathscr{L}_{\iota_\ell\xi})^{\mathrm{ss}}[\pi^\infty] = \dim \mathrm{H}^i_{\text{ét,c}}(\mathrm{Sh}, \mathscr{L}_{\iota_\ell\xi})[\pi^\infty]$.

Proof. The isomorphisms follow from Franke's spectral sequence [Fra98, Theorem 19] and the last assertion follows from Borel–Casselman's decomposition of $\mathrm{H}^i_{(2)}(\mathrm{Sh}, \mathscr{L}_\xi)$ as direct sums of certain multiplicities of π_f for each $\pi \in L^2_{\mathrm{disc}}(\mathbf{G}(\mathbf{A}_F))^{\mathrm{sm}}$ (thus $\mathrm{H}^i_{(2)}(\mathrm{Sh}, \mathscr{L}_\xi)$ is semisimple as a $\mathbf{G}(\mathbf{A}_{F,f})$-module); see [BC83]. The proof is the same as that of [KS23, Lemma 8.1(1)], thus omitted here. ☐

Theorem 4.5.6. *For all but finitely many finite places $\mathfrak{p}$ of F_1 not lying over ℓ or places in Σ, and for all sufficiently large positive integers j (depending on $\mathfrak{p}$),*

$$\mathrm{tr}\left(\sigma_{\mathfrak{p}}^j | \rho^\pi_{\mathrm{Sh}}\right) = m(\pi) \cdot \|\mathfrak{p}\|^{\frac{\dim_{\mathbb{C}}(\mathbf{X})}{2}\cdot j} \cdot \mathrm{tr}\left(\iota_\ell \tilde{\phi}^{\mathrm{GL}}_{\pi_{\mathfrak{p}_\flat}}(\sigma_{\mathfrak{p}}^j)\right),$$

where $\tilde{\phi}_{\pi_{\mathfrak{p}_\flat}}$ is the classical L-parameter of $\pi_{\mathfrak{p}_\flat}$. Moreover, the only nonzero term in the definition of ρ^π_{Sh} (see Definition 4.5.4*) appears in the middle degree $\dim_{\mathbb{C}}(\mathbf{X})$. In particular, ρ^π_{Sh} is a genuine representation of Gal_{F_1}.*

Proof. We imitate the arguments of [Kot92] and [KS23, Proposition 8.2]. Consider the test function $f = f_\infty \otimes f_{\Sigma^{\mathrm{St}}} \otimes f^{\Sigma^{\mathrm{St}} \cup \Sigma_F^\infty}$ on $\mathbf{G}(\mathbf{A}_F)$ such that

- $f_\infty = (\#\Pi_\xi(\mathbf{G}(F\otimes\mathbb{R})))^{-1} \sum_{\tau_\infty \in \Pi_\xi(\mathbf{G}(F\otimes\mathbb{R}))} f_{\tau_\infty}$, i.e., the average of the pseudo-coefficients for the discrete series L-packet of $\mathbf{G}(F\otimes\mathbb{R})$ associated with ξ.
- $f^{\Sigma^{\mathrm{St}} \cup \Sigma_F^\infty} \in \tilde{\mathcal{H}}\left(\mathbf{G}(\mathbf{A}_{F,f}^{\Sigma^{\mathrm{St}}})\right)$ so that, on the finite set of cuspidal automorphic representations $\dot{\pi}$ satisfying
$$(\dot{\pi}_f)^{\mathcal{K}} \neq 0 \qquad \text{and} \qquad \mathrm{tr}(f_\infty | \dot{\pi}_\infty) \neq 0,$$
it acts by the scalar 1 on those $\dot{\pi}$ satisfying
$$\dot{\pi}^\Sigma \text{ has the same } \tilde{\mathbb{T}}^\Sigma\text{-character as } \pi^\Sigma$$

and

$$\tilde{\dot{\pi}}_v \cong \tilde{\pi}_v \qquad (v \in \Sigma \smallsetminus \Sigma^{\mathrm{St}}),$$

and by the scalar 0 on the remaining representations in this finite set. Let $\Sigma_{\mathrm{aux}} \subset \Sigma_F^{\mathrm{fin}} \smallsetminus \Sigma^{\mathrm{St}}$ be the finite set of places at which the local component of $f^{\Sigma^{\mathrm{St}} \cup \Sigma_F^\infty}$ is not the unit of the corresponding spherical Hecke algebra.

- $f_{\Sigma^{\mathrm{St}}} = \otimes_{v \in \Sigma^{\mathrm{St}}} f^{\mathbf{G}}_{\mathrm{Lef},v}$, where each $f^{\mathbf{G}}_{\mathrm{Lef},v}$ is a Lefschetz function (see Definition A.4.7).

Choose a finite place $\mathfrak{p}$ of F_1 whose underlying place $\mathfrak{p}_\flat$ of F does not belong to $\Sigma \cup \Sigma^{\mathrm{aux}}$ and whose residue characteristic p is different from ℓ. Choose an isomorphism $\iota_p : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}_p}$ such that $\iota_p \circ \tau_0 : F_1 \to \overline{\mathbb{Q}_p}$ induces the place $\mathfrak{p}$. Then the stabilized Langlands–Kottwitz formula (Theorem 4.5.3) simplifies to

$$\sum_{i=0}^{2\dim_{\mathbb{C}}(\mathbf{X})} (-1)^i \iota_\ell^{-1} \mathrm{Tr}\left(\iota_\ell f^\infty \sigma_{\mathfrak{p}}^j | \mathrm{H}^i_{\text{ét,c}}(\mathrm{Sh}, \mathscr{L}_{\iota_\ell \xi})\right) = \mathrm{ST}^{\mathbf{G}^*}_{\mathrm{ell},\chi}(h^{\mathbf{G}^*}_{\xi,j}),$$

because the stable orbital integrals of $h_v^{\mathbf{G}^{\mathfrak{e}}}$ vanish for $\mathbf{G}^{\mathfrak{e}} \neq \mathbf{G}^*$ as they equal κ-orbital integrals of $f^{\mathbf{G}}_{\mathrm{Lef},v}$ with $\kappa \neq \mathbf{1}$ up to a nonzero constant, and $f^{\mathbf{G}}_{\mathrm{Lef},v}$ is stabilizing (see Definition A.4.7); see [Lab99, Theorem 4.3.4]. Note that the left-hand side equals

$$(-1)^{\dim_{\mathbb{C}}(\mathbf{X}) + \sum_{v \in \Sigma^{\mathrm{St}}} q(\mathbf{G}_v)} \mathrm{tr}\left(\sigma_{\mathfrak{p}}^j | \rho^\pi_{\mathrm{Sh}}\right)$$

by definition of $f^\infty = f_{\Sigma^{\mathrm{St}}} \otimes f^{\Sigma^{\mathrm{St}} \cup \Sigma_F^\infty}$, where $q(\mathbf{G}_v)$ is the F_v-rank of $\mathbf{G}_v$.

Then it follows from the simple stable trace formula (Theorem A.4.10) and Definition A.4.8 that

$$\mathrm{ST}^{\mathbf{G}^*}_{\mathrm{ell},\chi}(h^{\mathbf{G}^*}_{\xi,j}) = \mathrm{T}^{\mathbf{G}}_{\mathrm{cusp},\chi}(f^{p\infty} f'_{p,j} f_\infty) = \sum_{\dot{\pi} \in \mathrm{Irr}^{\mathrm{cusp}}_\chi(\mathbf{G})} m(\dot{\pi}) \mathrm{tr}(f^{p\infty} | \dot{\pi}^{p\infty}) \mathrm{tr}(f'_{p,j} | \dot{\pi}_p) \mathrm{tr}(f_\infty | \dot{\pi}_\infty) \tag{4.8}$$

where $f'_{p,j} = h^{\mathbf{G}^*}_{p,j}$ via the identification $\varrho_p : \mathbf{G}^*(F \otimes \mathbb{Q}_p) \xrightarrow{\sim} \mathbf{G}(F \otimes \mathbb{Q}_p)$. The term on the right-hand side vanishes unless $\dot{\pi}_p$ is unramified and both traces $\mathrm{tr}(f^{p\infty} | \dot{\pi}^{p\infty})$ and $\mathrm{tr}(f_\infty | \dot{\pi}_\infty)$ are nonzero. If a summand indexed by $\dot{\pi}$ contributes, then the choice of $f^{\Sigma^{\mathrm{St}} \cup \Sigma_F^\infty}$ implies that $\dot{\pi}^\Sigma$ has the same $\tilde{\mathbb{T}}^\Sigma$-character as π^Σ, and that

$$\tilde{\dot{\pi}}_v \cong \tilde{\pi}_v \qquad (v \in \Sigma \smallsetminus \Sigma^{\mathrm{St}}).$$

By strong multiplicity one for the strong functorial transfers and Corollary 4.2.2, $\dot{\pi}$ and π have the same local classical L-parameter at every finite place. At $v \in \Sigma^{\mathrm{St}}$ the packet is a singleton, and at $v \notin \Sigma$ both representations are spherical. Hence $\dot{\pi} \in [\pi]$. In particular, $\dot{\pi}_v \cong \pi_v$ for $v \in \Sigma_F(\{p\})$ as π_v is unramified. By Definition A.4.8, the right-hand side of Equation (4.8) equals

$$\begin{aligned} &\frac{1}{\#\Pi_\xi(\mathbf{G}(F \otimes \mathbb{R}))} \sum_{\dot{\pi} \in [\pi]} m(\dot{\pi}) (-1)^{\sum_{v \in \Sigma^{\mathrm{St}}} q(\mathbf{G}_v)} \mathrm{ep}(\dot{\pi}_\infty \otimes \xi) \mathrm{tr}\Big(h^{\mathbf{G}^*}_{p,j} | \prod_{v \in \Sigma_F(\{p\})} \pi_v\Big) \\ &= (-1)^{\dim_{\mathbb{C}}(\mathbf{X}) + \sum_{v \in \Sigma^{\mathrm{St}}} q(\mathbf{G}_v)} m(\pi) \mathrm{tr}\Big(h^{\mathbf{G}^*}_{p,j} | \prod_{v \in \Sigma_F(\{p\})} \pi_v\Big). \end{aligned}$$

Here we use that $m(\dot{\pi}) = m(\pi)$ for each $\dot{\pi} \in [\pi]$ by Arthur's multiplicity formula. Thus

$$\mathrm{tr}\left(\sigma_{\mathfrak{p}}^j | \rho^\pi_{\mathrm{Sh}}\right) = m(\pi) \mathrm{tr}\Big(h^{\mathbf{G}^*}_{p,j} | \prod_{v \in \Sigma_F(\{p\})} \pi_v\Big).$$

To analyze the right-hand side, consider the conjugacy class

$$\{\mu\} : \mathrm{GL}_{1,\mathbb{C}} \to \left(\mathrm{Res}_{F/\mathbb{Q}} \mathbf{G}\right)_{\mathbb{C}}$$

of Hodge cocharacters defined in §4.4. The irreducible highest-weight representation

$$\left(\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G}\right)^{\wedge} \cong \prod_{\tau\in\Sigma_F^{\infty}} \widehat{\mathbf{G}_{\tau}}$$

associated with $\{\mu\}$ is the standard representation on the τ_0 component and trivial on other components. Furthermore, the Satake parameter of π_v belongs to the σ_v-coset of $\left((\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G})\otimes\mathbb{Q}_p\right)^{\wedge}$, identified with $\left(\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G}\right)^{\wedge}$ via $\widehat{\iota_p}$, in the L-group. Thus the τ_0-component of the Satake parameter of the representation $\prod_{v\in\Sigma_F(\{p\})}\pi_v$ of $\left(\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G}\right)\otimes\mathbb{Q}_p$ is identified with the Satake parameter of $\pi_{\mathfrak{p}_\flat}$, since the composite

$$F \xrightarrow{\tau_0} \mathbb{C} \xrightarrow{\iota_p} \overline{\mathbb{Q}_p}$$

induces the place $\mathfrak{p}_\flat$. Then it follows from [Kot84, (2.2.1)] that

$$\mathrm{tr}\left(h_{p,j}^{\mathbf{G}^*}|\prod_{v\in\Sigma_F(\{p\})}\pi_v\right) = \|\mathfrak{p}\|^{\frac{\dim_{\mathbb{C}}(\mathbf{X})}{2}j}\,\mathrm{tr}\left(\iota_\ell\tilde{\phi}^{\mathrm{GL}}_{\pi_{\mathfrak{p}_\flat}}(\sigma_{\mathfrak{p}}^j)\right),$$

and the first assertion follows.

For every i, define the genuine semisimple Galois representation

$$V_i := \bigoplus_{\dot{\pi}\in[\pi]/\sim}\left(\mathrm{IH}^i\left(\mathrm{Sh},\mathscr{L}_{\iota_\ell\xi}\right)[\dot{\pi}^\infty]\right)^{\mathrm{ss}}.$$

By Lemma 4.5.5 and (4.7), we obtain

$$[\rho^{\pi}_{\mathrm{Sh}}] = \sum_{i=0}^{2\dim_{\mathbb{C}}(\mathbf{X})}(-1)^{i-\dim_{\mathbb{C}}(\mathbf{X})}[V_i].$$

The coefficient local system $\mathscr{L}_{\iota_\ell\xi}$ is pure of weight 0. Hence V_i is pure of weight i, by Pink's purity result [Pin92, Proposition 5.6.2] and the purity of intersection cohomology. (Note that the weight cocharacter of the Shimura datum which appears in [Pin92, §5.4] is trivial because $Z(\mathbf{G})$ is anisotropic).

Fix a place $\mathfrak{p}$ for which the first assertion of the theorem holds. Since $\dot{\pi}_{\mathfrak{p}_\flat}$ is tempered for every $\dot{\pi}\in[\pi]$, the first part of the theorem and Theorem 2.3.1 imply that, for all sufficiently large j,

$$\sum_{i=0}^{2\dim_{\mathbb{C}}(\mathbf{X})}(-1)^{i-\dim_{\mathbb{C}}(\mathbf{X})}\iota_\ell^{-1}\mathrm{tr}\left(\sigma_{\mathfrak{p}}^j|V_i\right)$$

is a power sum all of whose bases have complex absolute value $\|\mathfrak{p}\|^{\frac{j\dim_{\mathbb{C}}(\mathbf{X})}{2}}$.

Since V_i is pure of weight i and all constituents of a fixed V_i occur with the same sign, no cancellation is possible for $i\neq\dim_{\mathbb{C}}(\mathbf{X})$. The linear independence of the sequences $j\mapsto\alpha^j$ implies that $V_i=0$ for $i\neq\dim_{\mathbb{C}}(\mathbf{X})$. Thus only the middle degree contributes, and

$$\rho^{\pi}_{\mathrm{Sh}}\cong V_{\dim_{\mathbb{C}}(\mathbf{X})}$$

is a genuine representation of Gal_{F_1}. □

Corollary 4.5.7. *The Galois representation ρ^{π}_{Sh} has contribution only coming from the middle degree $\dim_{\mathbb{C}}(\mathbf{X})$, and for each $w\in\Sigma_{F_1}^{\mathrm{fin}}$ with underlying place $v\in\Sigma_F^{\mathrm{fin}}$ it has a subquotient whose semisimplification is isomorphic to*

$$\left(\rho_{\pi,\ell}|W_{F_{1,w}}\right)^{\mathrm{ss}}\otimes\iota_\ell\left|-\right|_{F_{1,w}}^{\frac{N(\mathbf{G})-1-\dim_{\mathbb{C}}(\mathbf{X})}{2}}\cong\iota_\ell\left(\tilde{\phi}^{\mathrm{GL}}_{\pi_v}\otimes\left|-\right|_{F_{1,w}}^{-\frac{\dim_{\mathbb{C}}(\mathbf{X})}{2}}\right)$$

as a $W_{F_{1,w}}$-module, where $\rho_{\pi,\ell}$ is defined in Theorem 4.3.6.

Proof. This follows from Theorem 4.5.6, together with the local-global compatibility and irreducibility of $\rho_{\pi,\ell}$ (Theorem 4.3.6), the Brauer–Nesbitt theorem, and the Chebotarev density theorem. $\square$

To end this section, we recall a hypothesis describing the Galois representation appearing in the generic part of the middle degree ℓ-adic cohomology of orthogonal and unitary Shimura varieties. It will later be used in the proof of the strong Kottwitz conjecture.

Hypothesis 4.5.8. Suppose ξ is the trivial representation of $(\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G}) \otimes_{\mathbb{Q}} \mathbb{C}$, except in Case O2, where its highest weight is

$$(1, \ldots, 1)$$

at every infinite place. Suppose π is a cuspidal automorphic representation of $\mathbf{G}(\mathbf{A}_F)$ that is cohomological for ξ, with generic elliptic global parameter

$$\boldsymbol{\psi} = \Pi_1 + \cdots + \Pi_r,$$

where each Π_j is a conjugate self-dual cuspidal automorphic representation of $\mathrm{GL}_{n_j}(\mathbf{A}_{F_1})$ for each $1 \leq j \leq r$, and $n_1 + \cdots + n_r = N(\mathbf{G})$. We set

$$\mathfrak{I} := \begin{cases} \{N(\mathbf{G}), N(\mathbf{G}) - 2, \ldots, 2, -2, \ldots, -N(\mathbf{G})\} & \text{in case O2,} \\ \{N(\mathbf{G}) - 1, N(\mathbf{G}) - 3, \ldots, 3 - N(\mathbf{G}), 1 - N(\mathbf{G})\} & \text{otherwise}. \end{cases}$$

There is a unique partition $\mathfrak{I} = \mathfrak{I}_1 \coprod \cdots \coprod \mathfrak{I}_r$ such that, for every j,

$$\phi_{\Pi_{j,\tau_0}}|_{\mathbb{C}^\times} \cong \bigoplus_{i \in \mathfrak{I}_j} \arg^i.$$

The Galois representation ρ^{π}_{Sh} has contribution only coming from the middle degree $\dim_{\mathbb{C}}(\mathbf{X})$, by the same argument as in the last part of the proof of Theorem 4.5.6. If $\rho^{\pi}_{\mathrm{Sh}} \neq 0$, then π_{τ_0} is the discrete series representation of $\mathbf{G}_{\tau_0}(\mathbb{R})$ that corresponds to a character of $\mathfrak{S}_{\tilde{\psi}_{\tau_0}}$. Let χ_π be the pullback of this character along the localization map

$$\mathfrak{S}_{\boldsymbol{\psi}} \to \mathfrak{S}_{\tilde{\psi}_{\tau_0}}.$$

Identify the global component group $\mathfrak{S}_{\boldsymbol{\psi}}$ with the kernel of the homomorphism

$$(\mathbb{Z}/2\mathbb{Z})^{\mathfrak{I}} \to \mathbb{Z}/2\mathbb{Z} \colon \sum_{i \in \mathfrak{I}} a_i e_i \mapsto \sum_{i \in \mathfrak{I}} a_i n_i.$$

Let $1 \leq j(\pi) \leq r$ be an integer such that χ_π equals the projection of $(\mathbb{Z}/2\mathbb{Z})^{\mathfrak{I}}$ to the $j(\pi)$-th coordinate. In Case O2, two distinct coordinate characters may have the same restriction to $\mathfrak{S}_{\boldsymbol{\psi}}$, so $\mathfrak{J}(\pi)$ may have two elements. We let $\mathfrak{J}(\pi)$ be the subset of $[r]_+$ consisting of all such choices of $j(\pi)$, then there is an isomorphism of Galois representations

$$\rho^{\pi}_{\mathrm{Sh}} \cong \Bigg(\bigoplus_{j \in \mathfrak{J}(\pi)} \rho_{\Pi_j,\ell} \Bigg)^{\oplus m(\pi)} : \mathrm{Gal}_{F_1} \to \mathrm{GL}_{m(\pi)\sum_{j \in \mathfrak{J}(\pi)} n_j}(\overline{\mathbb{Q}_\ell}).$$

Here $\rho_{\Pi_j,\ell}$ is the Galois representation attached to Π_j, for example as in [BLGGT14, Theorem 2.1.1][10].

Remark 4.5.9. In Case O, if $F = \mathbb{Q}$, then Hypothesis 4.5.8 is a corollary of [Zhu24, Corollary 9.8.10]. In general, Hypothesis 4.5.8 will be established in a sequel to [KSZ21], assuming the full endoscopic classification for the corresponding unitary and orthogonal groups, cf. [LL21, Hypothesis 6.6, Remark 6.7]. Note that the generic part of the endoscopic classification for such groups is established by Chen-Zou [CZ25, Theorem 2.6].

[10] In that reference, π is assumed to be a regular C-algebraic cuspidal essentially self-dual representation of GL_n, but the Galois representation is associated with $\pi \otimes |\det|^{(1-n)/2}$, which is regular L-algebraic.

5. Local and global Shimura varieties

In this section, we relate local shtuka spaces with minuscule μ (or local Shimura varieties) to global Shimura varieties, in order to prove a key result on the cohomology of local shtuka spaces, namely Corollary 5.2.3, using global methods.

Let p be a rational prime, let $\iota_p : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}_p}$ be a fixed isomorphism, and let K be a finite extension of $\mathbb{Q}_p$. Let $\ell \neq p$ be a rational prime and fix an isomorphism $\iota_\ell : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}_\ell}$. The isomorphism ι_ℓ fixes a square root of p in $\overline{\mathbb{Q}_\ell}$. The latter is an ℓ-adic unit and therefore has a reduction in $\overline{\mathbb{F}_\ell}^\times$.

5.1. **Basic uniformization of Shimura varieties.** In this subsection we briefly review the basic uniformization of the generic fiber of a Shimura variety following [Han20, §3.1] and [Ham25, §4]. We adopt the notation on local shtuka spaces from §3.2.

Let $\mathbb{G}$ be a connected reductive group over $\mathbb{Q}$ and $(\mathbb{G}, \mathbb{X})$ a Shimura datum of Abelian type with associated conjugacy class of Hodge cocharacters $\{\mu\} : \mathrm{GL}_{1,\mathbb{C}} \to \mathbb{G}_{\mathbb{C}}$. Suppose $\mathsf{G} = \mathbb{G} \otimes \mathbb{Q}_p$ is unramified, and we fix a Borel pair (B, T) for G, then we get from $\{\mu\}$ and ι_p a dominant cocharacter μ for $\mathsf{G}_{\overline{\mathbb{Q}_p}}$, with reflex field $E_\mu/\mathbb{Q}_p$.

For each neat compact open subgroup $\mathcal{K} = \mathcal{K}_p\mathcal{K}^p \leq \mathbb{G}(\mathbf{A}_f)$, we have the adic space $\mathcal{S}_{\mathcal{K}_p\mathcal{K}^p}(\mathbb{G}, \mathbb{X})$ over $\mathrm{Spa}(E_\mu)$ associated with the Shimura variety $\mathrm{Sh}_{\mathcal{K}_p\mathcal{K}^p}(\mathbb{G}, \mathbb{X})$, and we define

$$\mathcal{S}_{\mathcal{K}^p}(\mathbb{G}, \mathbb{X}) := \varprojlim_{\mathcal{K}_p \to \mathbf{1}} \mathcal{S}_{\mathcal{K}_p\mathcal{K}^p}(\mathbb{G}, \mathbb{X}),$$

which is representable by a perfectoid space because $(\mathbb{G}, \mathbb{X})$ is of Abelian type, see [Sch15, She17]. By the result of [Han20], there exists a canonical $\mathsf{G}(\mathbb{Q}_p)$-equivariant Hodge–Tate period map defined in [Sch15, CS17]

$$\pi_{\mathrm{HT}} : \mathcal{S}_{\mathcal{K}^p}(\mathbb{G}, \mathbb{X}) \to \mathrm{Gr}_{\mathsf{G},\mu},$$

where $\mathrm{Gr}_{\mathsf{G},\mu}$ is the Schubert cell of the $\mathbf{B}^+_{\mathrm{dR}}$-affine Grassmannian of [SW20] indexed by μ, defined over $\mathrm{Spa}(E_\mu)$.

We write $\mu^\bullet = -w_0(\mu) \in X_\bullet(\mathsf{G})$, where w_0 is the longest Weyl group element. Let $\mathsf{b} \in B(\mathsf{G}, \mathbf{1}, \mu^\bullet)_{\mathrm{bas}}$ be the unique basic element and $\mathbf{1} \in B(\mathsf{G})_{\mathrm{bas}}$ be the trivial element, then we obtain the open basic Newton stratum $\mathrm{Gr}^{\mathsf{b}}_{\mathsf{G},\mu} \subset \mathrm{Gr}_{\mathsf{G},\mu}$ as defined in [CS17, §3.5], and let $\mathcal{S}_{\mathcal{K}^p}(\mathbb{G}, \mathbb{X})^{\mathsf{b}} \subset \mathcal{S}_{\mathcal{K}^p}(\mathbb{G}, \mathbb{X})$ be the preimage of $\mathrm{Gr}^{\mathsf{b}}_{\mathsf{G},\mu}$ under π_{HT}, called the basic Newton stratum.

Suppose $p > 2$ and G is unramified, then Shen's result [She20, Theorem 1.2] together with [Han20, Proposition 3.1] state that $(\mathbb{G}, \mathbb{X})$ has a basic uniformization at p in the following sense:

Theorem 5.1.1. *There exists a $\mathbb{Q}$-inner form $\mathbb{G}'$ of $\mathbb{G}$ such that*

- $\mathbb{G}' \otimes \mathbf{A}^p_f \cong \mathbb{G} \otimes \mathbf{A}^p_f$ *as algebraic groups over* $\mathbf{A}^p_f$,
- $\mathbb{G}' \otimes \mathbb{Q}_p \cong \mathsf{G}_{\mathsf{b}}$,
- $\mathbb{G}'(\mathbb{R})$ *is compact modulo center,*

together with a $\mathbb{G}(\mathbf{A}_f)$-equivariant isomorphism of diamonds over $\breve{E} := \breve{\mathbb{Q}}_p E_\mu$:

$$\varprojlim_{\mathcal{K}^p \to \mathbf{1}} \mathcal{S}_{\mathcal{K}^p}(\mathbb{G}, \mathbb{X})^{\mathsf{b}} \cong \left(\underline{\mathbb{G}'(\mathbb{Q})\backslash\mathbb{G}'(\mathbf{A}_f)} \times_{\mathrm{Spd}(\breve{E})} \mathrm{Sht}(\mathsf{G}, \mathsf{b}, \mathbf{1}, \mu^\bullet)\right) / \underline{\mathsf{G}_{\mathsf{b}}(\mathbb{Q}_p)},$$

where $\mathsf{G}_{\mathsf{b}}(\mathbb{Q}_p)$ acts diagonally and $\mathbb{G}(\mathbf{A}_f) \cong \mathbb{G}'(\mathbf{A}^p_f) \times \mathsf{G}(\mathbb{Q}_p)$ acts on the right-hand side via the natural action of $\mathbb{G}'(\mathbf{A}^p_f)$ on $\mathbb{G}'(\mathbb{Q})\backslash\mathbb{G}'(\mathbf{A}_f)$ and the action of $\mathsf{G}(\mathbb{Q}_p)$ on $\mathrm{Sht}(\mathsf{G}, \mathsf{b}, \mathbf{1}, \mu^\bullet)$. Moreover, under the identification $\mathrm{Gr}^{\mathsf{b}}_{\mathsf{G},\mu} \cong \mathrm{Sht}(\mathsf{G}, \mathsf{b}, \mathbf{1}, \mu^\bullet)/\underline{\mathsf{G}_{\mathsf{b}}(\mathbb{Q}_p)}$, the Hodge–Tate period map:

$$\pi_{\mathrm{HT}} : \varprojlim_{\mathcal{K}^p \to \mathbf{1}} \mathcal{S}_{\mathcal{K}^p}(\mathbb{G}, \mathbb{X})^{\mathsf{b}} \to \mathrm{Gr}^{\mathsf{b}}_{\mathsf{G},\mu}$$

identifies with the natural projection

$$\left(\underline{\mathbb{G}'(\mathbb{Q})\backslash\mathbb{G}'(\mathbf{A}_f)} \times_{\mathrm{Spd}(\breve{E})} \mathrm{Sht}(\mathsf{G}, \mathsf{b}, \mathbf{1}, \mu^\bullet)\right) / \underline{\mathsf{G}_{\mathsf{b}}(\mathbb{Q}_p)} \to \mathrm{Sht}(\mathsf{G}, \mathsf{b}, \mathbf{1}, \mu^\bullet)/\underline{\mathsf{G}_{\mathsf{b}}(\mathbb{Q}_p)}.$$

This basic uniformization at p will allow us to deduce an isomorphism

$$\mathrm{R}\Gamma_c(\mathsf{G},\mathsf{b},\mathbf{1},\mu^\bullet)\otimes\iota_\ell\,|-|_{E_\mu}^{\frac{-\dim_{\mathbb{C}}(\mathbb{X})}{2}}[\dim_{\mathbb{C}}(\mathbb{X})]\otimes^{\mathrm{L}}_{\mathsf{G}_\mathsf{b}(\mathbb{Q}_p)}\mathcal{A}\left(\mathbb{G}'(\mathbb{Q})\backslash\mathbb{G}'(\mathbf{A}_f)/\mathcal{K}^p,\mathscr{L}_{\iota_\ell\xi}\right)$$
$$\cong\mathrm{R}\Gamma_c\left(\mathcal{S}_{\mathcal{K}^p}(\mathbb{G},\mathbb{X})^{\mathsf{b}},\mathcal{L}_{\iota_\ell\xi}\right)$$

of $\mathsf{G}(\mathbb{Q}_p)\times W_{E_\mu}$-modules, for each algebraic representation ξ of $\mathbb{G}_{\mathbb{C}}$ that is trivial on Z_{ac} (see Definition 4.5.2), where $\mathcal{L}_{\iota_\ell\xi}$ is the rigid analytification of the lisse $\overline{\mathbb{Q}_\ell}$-sheaf $\mathscr{L}_{\iota_\ell\xi}$ associated with ξ (see Definition 4.5.2). When composed with the morphism

$$\mathrm{R}\Gamma_c\left(\mathcal{S}_{\mathcal{K}^p}(\mathbb{G},\mathbb{X})^{\mathsf{b}},\mathcal{L}_{\iota_\ell\xi}\right)\to\mathrm{R}\Gamma_c\left(\mathcal{S}_{\mathcal{K}^p}(\mathbb{G},\mathbb{X}),\mathcal{L}_{\iota_\ell\xi}\right)$$

coming from excision with $\mathbb{Z}_\ell$-coefficients with respect to the open basic stratum $\mathcal{S}(\mathbb{G},\mathbb{X})^{\mathsf{b}}_{\mathcal{K}^p}\subset\mathcal{S}(\mathbb{G},\mathbb{X})_{\mathcal{K}^p}$, this isomorphism gives us a uniformization map between cohomologies. We recall the following result from [Ham25, Proposition 4.1].

Proposition 5.1.2. *Suppose $p>2$, then there exists a $\mathsf{G}(\mathbb{Q}_p)\times W_{E_\mu}$-equivariant map*

$$\Theta:\mathrm{R}\Gamma_c(\mathsf{G},\mathsf{b},\mathbf{1},\mu^\bullet)\otimes\iota_\ell\,|-|_{E_\mu}^{\frac{-\dim_{\mathbb{C}}(\mathbb{X})}{2}}[\dim_{\mathbb{C}}(\mathbb{X})]\otimes^{\mathrm{L}}_{\mathsf{G}_\mathsf{b}(\mathbb{Q}_p)}\mathcal{A}\left(\mathbb{G}'(\mathbb{Q})\backslash\mathbb{G}'(\mathbf{A}_f)/\mathcal{K}^p,\mathscr{L}_{\iota_\ell\xi}\right)$$
$$\to\mathrm{R}\Gamma_c\left(\mathcal{S}(\mathbb{G},\mathbb{X})_{\mathcal{K}^p},\mathcal{L}_{\iota_\ell\xi}\right)$$

functorial with respect to $\mathcal{K}^p$.

Next, we apply 'Boyer's trick,' an analogue of a result of [Boy99], relating the supercuspidal part of the cohomology of the local Shimura variety to the cohomology of the global Shimura variety. We recall the definition of being fully Hodge–Newton decomposable in the sense of [RV14, Definition 4.28] and [GHN19, Definition 3.1]:

Definition 5.1.3. Let G be a quasisplit reductive group over $\mathbb{Q}_p$ with a Borel pair (B,T), and let μ be a dominant cocharacter for $\mathsf{G}_{\overline{\mathbb{Q}_p}}$. Then (G,μ) is called *fully Hodge–Newton decomposable* if for every non-basic φ_K-conjugacy class $\mathsf{b}\in B(\mathsf{G},\mathbf{1},\mu)$ (see (3.3)), there exists a proper standard Levi subgroup M of G, a dominant cocharacter μ_{M} of $\mathsf{M}_{\overline{\mathbb{Q}_p}}$ and an element $\mathsf{b}_{\mathsf{M}}\in B(\mathsf{M},\mathbf{1},\mu_{\mathsf{M}})$ such that μ_{M} is conjugate to μ under $\mathsf{G}(\overline{\mathbb{Q}_p})$-action, and b_{M} is mapped to b under the natural map $B(\mathsf{M})\to B(\mathsf{G})$.[11]

Example 5.1.4. Suppose

- $\mathsf{G}=\mathrm{Res}_{K/\mathbb{Q}_p}G^*$ where $K/\mathbb{Q}_p$ is unramified and G^* is a quasisplit reductive group over K defined in §2.1, moreover we assume that G^*_{ad} is geometrically simple,
- μ is the dominant cocharacter of

$$\left(\mathrm{Res}_{K/\mathbb{Q}_p}G^*\right)_{\overline{\mathbb{Q}_p}}\cong\prod_{v\in\mathrm{Hom}(K,\overline{\mathbb{Q}_p})}G^*_{\overline{K}}$$

  that equals μ_1 defined in (2.5) on one factor and trivial on the other factors,

then $(\mathrm{Res}_{K/\mathbb{Q}_p}G^*,\mu)$ is fully Hodge–Newton decomposable: By [GHN19, Theorem 3.3, Proposition 3.4], it suffices to show that $\{\mu\}$ is minute for $\mathrm{Res}_{K/\mathbb{Q}_p}G^*_{\mathrm{ad}}$ as defined in [GHN19, Definition 3.2]. Then it suffices to show that $\{\mu_1\}$ is minute for G^* by [GHN19, §3.4], and this follows from the classification in [GHN19, Theorem 3.5]; cf. [GHN19, §3.7].

Finally, we recall the following result of Hamann [Ham25, Proposition 4.4].

[11]For a fixed $\mathsf{b}\in B(\mathsf{G},\mathbf{1},\mu)$, the condition that b is Hodge–Newton decomposable with respect to some proper Levi in the sense of [GHN19, Definition 3.1] is exactly the condition that the local Shimura datum (b,μ) is Hodge–Newton reducible in the sense of [RV14, Definition 4.28].

Proposition 5.1.5. *If $p > 2$, G is unramified, and $(\mathsf{G}, \mu^\bullet)$ is fully Hodge–Newton decomposable, then the uniformization map Θ in* Proposition 5.1.2 *induces an isomorphism of W_{E_μ}-modules*

$$\Theta_{\mathrm{sc}} : \mathrm{R}\Gamma_c(\mathsf{G}, \mathsf{b}, \mathbf{1}, \mu^\bullet)_{\mathrm{sc}} \otimes \iota_\ell |-|_{E_\mu}^{\frac{-\dim_{\mathbb{C}}(\mathbb{X})}{2}} [\dim_{\mathbb{C}}(\mathbb{X})] \otimes^{\mathrm{L}}_{\mathsf{G}_\mathsf{b}(\mathbb{Q}_p)} \mathcal{A}\left(\mathbb{G}'(\mathbb{Q})\backslash \mathbb{G}'(\mathbf{A}_f)/\mathscr{K}^p, \mathscr{L}_{\iota_\ell \xi}\right) \xrightarrow{\sim} \mathrm{R}\Gamma_c\left(\mathcal{S}_{\mathscr{K}^p}(\mathbb{G}, \mathbb{X}), \mathcal{L}_{\iota_\ell \xi}\right)_{\mathrm{sc}}$$

on the summands where $\mathsf{G}(\mathbb{Q}_p)$ acts via a supercuspidal representation.

Proof. The proofs of [Ham25, Lemma 4.3, Proposition 4.4] go through verbatim. The key point is that the non-basic Newton strata of the flag varieties $\mathrm{Gr}^{\mathsf{b}}_{\mathsf{G},\mu}$ are all parabolically induced from Newton strata on flag varieties associated with properly contained Levi subgroups of G, thus do not contribute to the supercuspidal part of cohomology. □

5.2. **Globalization.** In this subsection, assume that $K/\mathbb{Q}_p$ is unramified. Let G be one of the quasisplit reductive groups over K defined in §2.1, and we adopt the notation there. By a simple application of Krasner's lemma and the weak approximation theorem (see, for example, [Art13, Lemma 6.2.1], [CZ25, Theorem F.1]), we may choose a CM or totally real number field $F_1 \subset \mathbb{C}$ with maximal totally real subfield $F \subset F_1$, together with distinct rational primes p, q, r, such that

- p, q are inert in F_1,
- $$(F_1 \otimes \mathbb{Q}_p)/(F \otimes \mathbb{Q}_p) \cong K_1/K$$
  as extensions,
- $r \geq N(G) + 2$, and
- r is inert in F, and moreover $F_1 \otimes \mathbb{Q}_r$ is a ramified quadratic extension of $F \otimes \mathbb{Q}_r$ in case U.

Write $\mathfrak{p}_1 = (p), \mathfrak{q}_1 = (q), \mathfrak{r}_1 = (r) \in \Sigma^{\mathrm{fin}}_{F_1}$ with underlying places $\mathfrak{p}, \mathfrak{q}, \mathfrak{r} \in \Sigma^{\mathrm{fin}}_F$, respectively, and let τ_0 denote the fixed embedding $F \hookrightarrow \mathbb{C}$. We fix a vector space $\mathbf{V}$ over F_1 equipped with a nondegenerate Hermitian c-sesquilinear form on $\mathbf{V}$ such that

- $(\mathbf{V}, \mathbf{G} = \mathrm{U}(\mathbf{V})^\circ)$ is standard indefinite (see Definition 4.4.1),
- The Hasse–Witt invariant of $\mathbf{V}$ is trivial outside $\{\mathfrak{q}\} \cup \Sigma^\infty_F$, and
- $\mathbf{V} \otimes_F F_\mathfrak{p}$ is isomorphic to V as a c-Hermitian space over $F_\mathfrak{p}$.

Adopt the notation from §4.1 for $\mathbf{V}$. In particular, we have reductive groups $\mathbf{G}, \mathbf{G}^*, \mathbf{G}^\sharp$ over F with $\mathbf{G}_{\mathrm{ad}}$ geometrically simple; if $F_1 \neq F$, we have fixed a totally imaginary element $\daleth \in F_1^\times$, so each embedding $\tau : F \to \mathbb{C}$ extends to an embedding $\tau : F_1 \to \mathbb{C}$ sending $\daleth$ to $\mathbb{R}_+\mathrm{i}$. Then $G \cong \mathbf{G}_\mathfrak{p}$.

Let $(\mathrm{Res}_{F/\mathbb{Q}} \mathbf{G}, \mathbf{X})$ be the Shimura datum defined in §4.4 with a conjugacy class of Hodge cocharacters $\{\mu\}$. Via ι_p, we may regard $\{\mu\}$ as a conjugacy class of cocharacters

$$\mathrm{GL}_{1,\overline{\mathbb{Q}_p}} \to \left(\mathrm{Res}_{F/\mathbb{Q}} \mathbf{G}\right)_{\overline{\mathbb{Q}_p}} = \prod_{v \in \mathrm{Hom}(F, \overline{\mathbb{Q}_p})} \mathbf{G} \otimes_{F,v} \overline{\mathbb{Q}_p}.$$

Then it contains a dominant cocharacter μ that equals μ_1 defined in (2.5) on one factor and trivial on the other factors. In particular, $\left(\mathrm{Res}_{K/\mathbb{Q}_p} G, \mu\right)$ is fully Hodge–Newton decomposable by Example 5.1.4.

Let $(\mathbf{V}', \mathbf{G}' = \mathrm{U}(\mathbf{V}')^\circ)$ be a standard definite pure inner twist of $\mathbf{G}^*$ (see Definition 4.4.1) such that the Hasse–Witt invariant of $\mathbf{V}'$ is nontrivial at $\mathfrak{p}$ and trivial outside $\{\mathfrak{p}, \mathfrak{q}\} \cup \Sigma^\infty_F$ (such a pure inner twist exists uniquely), and define $J := \mathbf{G}'_\mathfrak{p}$, which is isomorphic to G_{b_1}, where b_1 is the unique nontrivial basic element in $B(G)_{\mathrm{bas}}$. We also regard b_1 as the unique nontrivial basic element in $B(\mathrm{Res}_{K/\mathbb{Q}_p} G)_{\mathrm{bas}}$ via the isomorphism $B(\mathrm{Res}_{K/\mathbb{Q}_p} G)_{\mathrm{bas}} \cong B(G)_{\mathrm{bas}}$.

We have the following globalization result.

Proposition 5.2.1. *Suppose $\rho \in \Pi_{\mathrm{sc}}(J)$ is supercuspidal, then there exists:*

- *a control tuple $\bigstar$ for $\mathbf{G}^*$ (see* Definition 4.3.1*) such that $\mathfrak{p} \in \Sigma^{\mathrm{sc}}, \Sigma^\circ = \{\mathfrak{p}, \mathfrak{q}\}$ and $\Sigma^{\mathrm{St}} = \{\mathfrak{q}\}$.*

- *a* $\bigstar$*-split compact open subgroup* $\mathscr{K}^{\mathfrak{p}} \leq \mathbf{G}'(\mathbf{A}^{\mathfrak{p}}_{F,f})$ *(see* Definition 4.3.3*); and*
- *a* $\bigstar$*-good automorphic representation* $\Pi' = \otimes'_v \Pi'_v$ *of* $\mathbf{G}'(\mathbf{A}_F)$ *(see* Definition 4.3.4*) such that* $\Pi'_{\mathfrak{p}} \cong \rho$ *and* $(\Pi'^{\mathfrak{p}}_f)^{\mathscr{K}^{\mathfrak{p}}} \neq 0$*.*

Such a tuple $(\bigstar, \mathscr{K}^{\mathfrak{p}}, \Pi')$ *is called a good globalization of* ρ*.*

Proof. Choose a finite place $\mathfrak{r}$ of F not in $\{\mathfrak{p}, \mathfrak{q}\}$, and let $\Sigma^{\mathrm{sc}} := \{\mathfrak{p}, \mathfrak{r}\}$. It follows from [CZ25, Theorems C.3 and D.2] that there exists a supercuspidal representation $\pi_{\mathfrak{r}}$ of $\mathbf{G}'(F_{\mathfrak{r}})$ with a simple supercuspidal L-parameter. Since $\mathbf{G}'(F \otimes \mathbb{R})$ admits discrete series, the standard Plancherel density theorem, for example [Shi12, Theorem 1.1.(i)], applies. In particular, if we choose an algebraic irreducible representation ξ of $\left(\mathrm{Res}_{F/\mathbb{Q}} \mathbf{G}'\right) \otimes \mathbb{C}$, then it follows from [Shi12, Theorem 1.1.(i)] that there exists a ξ-cohomological cuspidal automorphic representations Π' of $\mathbf{G}'(\mathbf{A}_F)$ such that $\Pi'_{\mathfrak{q}}$ is Steinberg, $\Pi'_{\mathfrak{p}} \cong \rho$, and $\Pi'_{\mathfrak{r}} \cong \pi_{\mathfrak{r}}$.

By choosing a sufficiently large finite subset of places Σ of F containing $\{\mathfrak{p}, \mathfrak{q}, \mathfrak{r}\}$ and Σ^{∞}_F, we can make sure Π'_v is unramified for all $v \in \Sigma^{\mathrm{fin}}_F \smallsetminus \Sigma$. We can also choose a $\bigstar$-split compact open subgroup $\mathscr{K}^{\mathfrak{p}} \leq \mathbf{G}'(\mathbf{A}^{\mathfrak{p}}_{F,f})$ such that $(\Pi'^{\mathfrak{p}}_f)^{\mathscr{K}^{\mathfrak{p}}} \neq 0$. ☐

Suppose $\rho \in \Pi_{\mathrm{sc}}(J)$ is supercuspidal with supercuspidal classical L-parameter $\tilde{\phi}$, and let $(\bigstar, \mathscr{K}^{\mathfrak{p}}, \Pi')$ be a good globalization of ρ. Let

$$\phi^{\Sigma}_{\Pi'} : \tilde{\mathbb{T}}^{\Sigma} \to \mathbb{C},$$

be the Hecke character associated with Π', and set $\mathfrak{m} := \iota_\ell \ker(\phi^{\Sigma}_{\Pi'})$, which is a maximal ideal of $\iota_\ell \tilde{\mathbb{T}}^{\Sigma}$. Consider the uniformization map

$$\begin{aligned}\Theta : \mathrm{R}\Gamma_c(G, b_1, \mathbf{1}, \mu_1) \otimes \iota_\ell \, |{-}|_{E_\mu}^{\frac{-\dim_{\mathbb{C}}(\mathbf{X})}{2}} [\dim_{\mathbb{C}}(\mathbf{X})] \otimes^{\mathrm{L}}_{J(K)} \mathcal{A}(\mathbf{G}'(F)\backslash \mathbf{G}'(\mathbf{A}_{F,f})/\mathscr{K}^{\mathfrak{p}}; \mathscr{L}_{\iota_\ell \xi}) \\ \to \mathrm{R}\Gamma_c(\mathcal{S}_{\mathscr{K}^{\mathfrak{p}}}(\mathrm{Res}_{F/\mathbb{Q}} \mathbf{G}, \mathbf{X}), \mathcal{L}_{\iota_\ell \xi})\end{aligned}$$

which is $G(K) \times W_{K_1}$-equivariant and functorial with respect to $\mathscr{K}^{\mathfrak{p}}$ by Proposition 5.1.2. Here we use that $\mathrm{Sht}(\mathrm{Res}_{K/\mathbb{Q}_p} G, b_1, \mathbf{1}, \mu^{\bullet})$ is naturally isomorphic to $\mathrm{Sht}(G, b_1, \mathbf{1}, \mu_1)$; see §3.2. We localize both sides at $\mathfrak{m}$ and restrict to the parts on both sides where $G(K)$ acts via supercuspidal representations to get an isomorphism

$$\begin{aligned}\Theta_{\mathfrak{m},\mathrm{sc}} : \mathrm{R}\Gamma_c(G, b_1, \mathbf{1}, \mu_1)_{\mathrm{sc}} \otimes \iota_\ell \, |{-}|_{E_\mu}^{\frac{-\dim_{\mathbb{C}}(\mathbf{X})}{2}} [\dim_{\mathbb{C}}(\mathbf{X})] \otimes^{\mathrm{L}}_{J(K)} \mathcal{A}(\mathbf{G}'(F)\backslash \mathbf{G}'(\mathbf{A}_{F,f})/\mathscr{K}^{\mathfrak{p}}, \mathscr{L}_{\iota_\ell \xi})_{\mathfrak{m}} \\ \xrightarrow{\sim} \mathrm{R}\Gamma_c(\mathcal{S}_{\mathscr{K}^{\mathfrak{p}}}(\mathrm{Res}_{F/\mathbb{Q}} \mathbf{G}, \mathbf{X}), \mathcal{L}_{\iota_\ell \xi})_{\mathfrak{m}}\end{aligned}$$

by the strong multiplicity one result Corollary 4.2.2 and the basic uniformization result Proposition 5.1.5 and Example 5.1.4. Note that, by Corollary 4.2.2 again, the left-hand side decomposes as a direct sum

$$\bigoplus_{\dot{\Pi}'} \left(\mathrm{R}\Gamma_c(G, b_1, \mathbf{1}, \mu_1)_{\mathrm{sc}} \otimes \iota_\ell \, |{-}|_{E_\mu}^{\frac{-\dim_{\mathbb{C}}(\mathbf{X})}{2}} [\dim_{\mathbb{C}}(\mathbf{X})] \otimes^{\mathrm{L}}_{J(K)} \iota_\ell (\dot{\Pi}'_f)^{\mathscr{K}^{\mathfrak{p}}} \right)^{\oplus m(\dot{\Pi}')}, \tag{5.1}$$

where $\dot{\Pi}'$ runs through $\bigstar$-good automorphic representations of $\mathbf{G}'(\mathbf{A}_F)$ such that $(\dot{\Pi}')^{\Sigma}$ has associated Hecke character $\phi^{\Sigma}_{\Pi'}$ and $\dot{\Pi}'_{\mathfrak{p}}$ has classical L-parameter $\tilde{\phi}$.

Corollary 5.2.2. *Both sides of* $\Theta_{\mathfrak{m},\mathrm{sc}}$ *are concentrated in cohomological degree* $\dim_{\mathbb{C}}(\mathbf{X})$*. Thus* $\Theta_{\mathfrak{m},\mathrm{sc}}$ *identifies their only nonzero cohomology groups as* $G(K) \times W_{K_1}$*-modules. Moreover, this* W_{K_1}*-module has a subquotient whose semisimplification is isomorphic to*

$$\iota_\ell \left(\tilde{\phi}^{\mathrm{GL}} \otimes |{-}|_{K_1}^{-\frac{\dim_{\mathbb{C}}(\mathbf{X})}{2}} \right).$$

Proof. By the direct sum decomposition (5.1), it suffices to prove the concentration statement for each summand

$$\mathrm{R}\Gamma_c(G, b_1, \mathbf{1}, \mu_1)_{\mathrm{sc}} \otimes \iota_\ell \, |-|_{E_\mu}^{-\frac{\dim_{\mathbb{C}}(\mathbf{X})}{2}} [\dim_{\mathbb{C}}(\mathbf{X})] \otimes^{\mathrm{L}}_{J(K)} \iota_\ell(\dot{\Pi}'_f)^{\mathscr{K}^{\mathfrak{p}}},$$

where $\dot{\Pi}'$ runs through $\bigstar$-good automorphic representations of $\mathbf{G}'(\mathbf{A}_F)$ such that $(\dot{\Pi}')^{\Sigma}$ has associated Hecke character $\phi^{\Sigma}_{\Pi'}$ and $\dot{\Pi}'_{\mathfrak{p}}$ has classical L-parameter $\tilde{\phi}$. Fix such a $\dot{\Pi}'$, and let $\dot{\Pi}$ be a $\bigstar$-good transfer of $\dot{\Pi}'$ to $\mathbf{G}$; see Theorem 4.3.5. Let

$$\rho_{\Pi',\ell} : \mathrm{Gal}_{F_1} \to \mathrm{GL}_{N(\mathbf{G})}(\overline{\mathbb{Q}_\ell})$$

be the Galois representation associated with Π' by Theorem 4.3.6. Since $(\dot{\Pi}')^{\Sigma}$ has associated Hecke character $\phi^{\Sigma}_{\Pi'}$, the same theorem and the strong multiplicity one result imply that $\rho_{\Pi',\ell}$ is also the Galois representation associated with $\dot{\Pi}'$ and to its transfer $\dot{\Pi}$. Under the uniformization isomorphism $\Theta_{\mathfrak{m},\mathrm{sc}}$, the above summand corresponds to the $\dot{\Pi}^{\Sigma}$-isotypic part of the target. By Corollary 4.5.7, this isotypic part occurs only in degree $\dim_{\mathbb{C}}(\mathbf{X})$. Hence every summand in (5.1) is concentrated in degree $\dim_{\mathbb{C}}(\mathbf{X})$, and therefore both sides of $\Theta_{\mathfrak{m},\mathrm{sc}}$ are concentrated in degree $\dim_{\mathbb{C}}(\mathbf{X})$.

It remains to prove the asserted existence of the W_{K_1}-subquotient. For this, take the particular summand corresponding to the original globalization $\dot{\Pi}' = \Pi'$, which occurs in (5.1). Let Π be a $\bigstar$-good transfer of Π' to $\mathbf{G}$. Applying Corollary 4.5.7 to the Π^{Σ}-isotypic part of the target gives a W_{K_1}-subquotient whose semisimplification is isomorphic to

$$\iota_\ell \left(\tilde{\phi}^{\mathrm{GL}} \otimes |-|_{K_1}^{-\frac{\dim_{\mathbb{C}}(\mathbf{X})}{2}} \right).$$

Transporting this subquotient through the isomorphism $\Theta_{\mathfrak{m},\mathrm{sc}}$ proves the claim. ∎

In particular, by the direct sum decomposition (5.1), we get the following key corollary:

Corollary 5.2.3. *If $\tilde{\phi} \in \tilde{\Phi}(J)$ is a supercuspidal L-parameter with associated packet $\tilde{\Pi}_{\tilde{\phi}}(J)$, then the direct summand of*

$$\bigoplus_{\tilde{\rho}' \in \tilde{\Pi}_{\tilde{\phi}}(J)} \mathrm{R}\Gamma_c(G, b_1, \mathbf{1}, \mu_1)[\iota_\ell \tilde{\rho}']$$

where $G(K)$ acts by supercuspidal representations, denoted by

$$\bigoplus_{\tilde{\rho}' \in \tilde{\Pi}_{\tilde{\phi}}(J)} \mathrm{R}\Gamma_c(G, b_1, \mathbf{1}, \mu_1)[\iota_\ell \tilde{\rho}']_{\mathrm{sc}},$$

is concentrated in the middle degree 0, and it has a subquotient whose semisimplification is isomorphic to $\iota_\ell \tilde{\phi}^{\mathrm{GL}}$ as a W_{K_1}-module.

6. Proof of the compatibility property

In this section we prove Theorem A. Here p is a rational prime, $K/\mathbb{Q}_p$ is an unramified finite extension, ℓ is a rational prime distinct from p with a fixed isomorphism $\iota_\ell : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}_\ell}$, and $(G = G^*_{b_0}, \varrho_{b_0}, z_{b_0})$ is an extended pure inner form of G^* of Case O or Case U as in §2, such that G splits over an unramified finite extension of K. Let $\pi \in \Pi(G)$ be an irreducible smooth representation with classical L-parameter $\tilde{\phi} \in \tilde{\Phi}(G^*)$. We will show $\tilde{\phi}^{\mathrm{ss}} = \tilde{\phi}^{\mathrm{FS}}_{\pi}|_{W_{K_1}}$ using induction on $n(G)$.

In Case O2 or U, when $n(G) = 1$, G is a torus. Thus the assertion is known by compatibility of Fargues–Scholze LLC with local class field theory, see Theorem 3.1.1.

In Case U, when $n(G) = 2$, choose a quaternion algebra D over K containing K_1 such that the Hermitian space V is identified with D, viewed as a two-dimensional right K_1-vector space with the Hermitian form induced by the reduced norm. G is contained in the unitary similitude group

$$G^\sharp := \mathrm{GU}(V) \cong (\mathrm{GL}_1(D) \times \mathrm{Res}_{K_1/K}\mathrm{GL}_1)/\mathrm{GL}_1,$$

where GL_1 acts anti-diagonally. By [Tad92, Proposition 2.2], for any $\pi \in \Pi(G)$, there exists $\pi^\sharp \in \Pi(G^\sharp)$ such that π is a subrepresentation of $\pi^\sharp|_{G(K)}$. So the assertion follows from compatibility for $G^\sharp$ (see [HL24, Lemma 4.13(3)]) and compatibility of Fargues–Scholze LLC with central extensions (see Theorem 3.1.1).

In Case O1, when $n(G) = 1$, G is of the form $\mathrm{PGL}_1(D)$ for some quaternion algebra D over K, and the LLC for G defined in Theorem 2.3.1 equals the LLC for G via the LLC for $\mathrm{GL}_1(D)$ constructed in [DKV84, Rog83] and the projection $\mathrm{GL}_1(D) \to \mathrm{PGL}_1(D)$; see [AG17, pp. 385–386]. Thus the main theorem follows from compatibility for inner forms of general linear groups [HKW22, Theorem 6.6.1] and compatibility of Fargues–Scholze LLC with central extensions; see Theorem 3.1.1.

In Case O2, when $n(G) = 2$, G is isomorphic to

$$\left(\mathrm{Res}_{K'/K}\,\mathrm{SL}_1(D)_{K'}\right)/\mu_2,$$

where K' is either $K \times K$ or the unique unramified quadratic extension of K, and D is a quaternion algebra over K; cf. [KR99, §0]. In fact we can prove the main theorem whenever K' is an étale extension of K of rank at most two. The L-parameter is constructed for G as follows (not only up to outer automorphism): For any $\pi \in \Pi(G)$, by [Tad92, Proposition 2.2] there exists $\pi^\sharp \in \Pi(G^\sharp)$, where

$$G^\sharp = \left(\mathrm{Res}_{K'/K}\mathrm{GL}_1(D)_{K'}\right)/\mathrm{GL}_1$$

is a group containing G, such that π is a subrepresentation of the restriction of $\pi^\sharp$ to $G(K)$. Then the L-parameter ϕ_π is given by $\phi_{\pi^\sharp}$ composed with the natural map ${}^LG^\sharp \to {}^LG$; see [AG17, pp. 385–386]. Thus, as before, the main theorem follows from compatibility for inner forms of general linear groups [HKW22, Theorem 6.6.1] and compatibility of Fargues–Scholze LLC with central extensions, Theorem 3.1.1.

Then, in all remaining cases G_{ad} is geometrically simple. We suppose throughout this section that the assertion is known for $G(n_0)$ for each $n_0 < n(G)$.

First, if π is non-supercuspidal, then the assertion is true:

Proposition 6.0.1. *If $\pi \in \Pi(G)$ is a subquotient of a parabolic induction, then $\tilde{\phi}_\pi^{\mathrm{ss}} = \tilde{\phi}_\pi^{\mathrm{FS}}|_{W_{K_1}}$.*

Proof. Suppose π is a subquotient of $\mathrm{I}_P^G(\sigma)$ where $P \leq G$ is a proper parabolic subgroup with Levi subgroup M, and $\sigma \in \Pi(M)$. By compatibility of Fargues–Scholze LLC with parabolic induction (see Theorem 3.1.1) and compatibility of classical LLC with parabolic induction (see Proposition 2.4.3), the assertion for π follows from the assertion for σ. The Levi subgroup M of G is of the form

$$G(n_0) \times \mathrm{Res}_{K_1/K}(H)$$

for some integer $n_0 < n(G)$, and H is a product of general linear groups. So the assertion follows from the induction hypothesis and compatibility of Fargues–Scholze LLC with products (see Theorem 3.1.1) and compatibility of Fargues–Scholze LLC with classical LLC for general linear groups (see Theorem 3.1.1). ☐

Assume now that π is supercuspidal, so $\tilde{\phi}$ is a discrete L-parameter by Theorem 2.3.1. There are two cases that can occur for the L-packet $\tilde{\Pi}_{\tilde{\phi}}(G^*)$:

(1) Case (1): $\tilde{\Pi}_{\tilde{\phi}}(G^*)$ consists entirely of supercuspidal representations,

(2) Case (2): $\tilde{\Pi}_{\tilde{\phi}}(G^*)$ contains a non-supercuspidal representation.

In the following subsections, we prove the main theorem in each case.

6.1. **The first case.** In Case (1), $\tilde{\Pi}_{\tilde{\phi}}(G^*)$ consists entirely of supercuspidal representations, so it follows from Corollary 2.5.2 that $\tilde{\phi}$ is a supercuspidal L-parameter.

We first prove the compatibility for representations of G_{b_1}, where b_1 is the unique nontrivial basic element in $B(G)_{\mathrm{bas}}$.

Proposition 6.1.1. *If $\tilde{\phi} \in \tilde{\Phi}_{\mathrm{sc}}(G^*)$ is a supercuspidal L-parameter, then $\tilde{\phi}^{\mathrm{FS}}_{\tilde{\rho}}|_{W_{K_1}} = \tilde{\phi}$ for any $\tilde{\rho} \in \tilde{\Pi}_{\tilde{\phi}}(G_{b_1})$.*

Proof. First assume that $\kappa_{b_0}(-1) = 1$, so $G \cong G^*$. Write

$$\tilde{\phi}^{\mathrm{GL}} = \phi_1 \oplus \cdots \oplus \phi_r,$$

where the ϕ_i are irreducible representations of W_{K_1}. By Corollary 5.2.3, Equation (3.5) and the compatibility of Fargues–Scholze LLC with contragredients, Theorem 3.1.1, we see that ϕ_i appears in

$$\bigoplus_{\tilde{\rho} \in \tilde{\Pi}_{\tilde{\phi}}(G_{b_1})} \mathrm{Mant}_{G,b_1,\mu_1}(\iota_\ell \tilde{\rho})$$

as a representation of W_{K_1}, for each i. It follows that ϕ_i is an irreducible subquotient of $(\tilde{\phi}^{\mathrm{FS}}_{\tilde{\rho}}|_{W_{K_1}})^{\mathrm{GL}}$ for each i and each $\tilde{\rho} \in \tilde{\Pi}_{\tilde{\phi}}(G_{b_1})$, by Corollary 3.4.3 and [Kos21, Theorem 1.3].[12] Since $\tilde{\phi}^{\mathrm{GL}}$ and $(\tilde{\phi}^{\mathrm{FS}}_{\tilde{\rho}}|_{W_{K_1}})^{\mathrm{GL}}$ are both semisimple with dimension $N(G)$, we deduce that they are equal. Thus it follows from [GGP12, Theorem 8.1.(ii)] that $\tilde{\phi} = \tilde{\phi}^{\mathrm{FS}}_{\tilde{\rho}}|_{W_{K_1}}$.

Now assume that $\kappa_{b_0}(-1) = -1$, so $G_{b_1} \cong G^*$. It follows from the weak Kottwitz conjecture Corollary 3.4.2 that some $\tilde{\pi} \in \tilde{\Pi}_{\tilde{\phi}}(G^*)$ appears in $\mathrm{Mant}_{G,b_1,\mu_1}(\iota_\ell \tilde{\rho})$. We then have $\tilde{\phi}^{\mathrm{FS}}_{\tilde{\pi}} = \tilde{\phi}^{\mathrm{FS}}_{\tilde{\rho}}$ by Proposition 3.2.3. Thus

$$\tilde{\phi}^{\mathrm{FS}}_{\tilde{\rho}}|_{W_{K_1}} = \tilde{\phi}^{\mathrm{FS}}_{\tilde{\pi}}|_{W_{K_1}} = \tilde{\phi},$$

where the second equality follows from the first case, upon replacing $(G = G^*_{b_0}, \varrho_{b_0}, z_{b_0})$ with $(G_{b_1}, \varrho_{b_0+b_1}, z_{b_0+b_1})$. ◻

Finally, to deduce the compatibility for $\tilde{\pi} \in \tilde{\Pi}_{\tilde{\phi}}(G)$, we apply this result with $(G = G^*_{b_0}, \varrho_{b_0}, z_{b_0})$ replaced by $(G_{b_1}, \varrho_{b_0+b_1}, z_{b_0+b_1})$, because then the unique nontrivial basic element in $B(G_{b_1})_{\mathrm{bas}}$ induces an inner form of G_{b_1} that is isomorphic to G.

6.2. **The second case.** In the second case where $\tilde{\Pi}_{\tilde{\phi}}(G^*)$ contains a non-supercuspidal representation, we write

$$\tilde{\phi}^{\mathrm{GL}} = \phi_1 \oplus \cdots \oplus \phi_k \oplus \phi_{k+1} \oplus \cdots \oplus \phi_r,$$

where the ϕ_i are irreducible representations of $W_{K_1} \times \mathrm{SL}_2(\mathbb{C})$ of dimension d_i for each i, and d_i is odd if and only if $i \leq k$.

We now prove the main theorem in this case.

Proposition 6.2.1. *If $\tilde{\Pi}_{\tilde{\phi}}(G^*)$ contains a non-supercuspidal representation $\tilde{\rho}^*_{\mathrm{nsc}}$, then $\tilde{\phi}^{\mathrm{FS}}_{\tilde{\pi}}|_{W_{K_1}} = \tilde{\phi}^{\mathrm{ss}}$ for every $\tilde{\pi} \in \tilde{\Pi}_{\tilde{\phi}}(G)$.*

Proof. Write $\tilde{\rho}^*_{\mathrm{nsc}} = \tilde{\pi}_{[I]}$ for some $I \in \mathscr{P}([r]_+)/\sim_k$ with $\#[I] \equiv 0 (\mathrm{mod}\ 2)$. We prove by induction on $m := \#J$ that

$$\tilde{\phi}^{\mathrm{FS}}_{\tilde{\pi}}|_{W_{K_1}} = \tilde{\phi}^{\mathrm{ss}}$$

for every

$$\tilde{\pi} = \tilde{\pi}_{[I \oplus J]} \in \tilde{\Pi}_{\tilde{\phi}}(G) \cup \tilde{\Pi}_{\tilde{\phi}}(G_{b_1}).$$

[12]Note that the notation $\mathcal{M}_{(G,b,\mu),K}$ in [Kos21] is just our $\mathrm{Sht}_{\mathscr{K}}(G, b, b_0, \mu^\bullet)$, so there exists no dual appearing; cf. [Ham25, Remark 3.8].

If $m = 0$, then $\tilde{\pi} = \tilde{\rho}^*_{\mathrm{nsc}}$, and the assertion follows from Proposition 6.0.1. Assume $m > 0$ and that the result is known for smaller values of m. Let $\tilde{\pi} = \tilde{\pi}_{[I\oplus J]}$ with $\#J = m$. If $\tilde{\pi}$ is not supercuspidal, the assertion again follows from Proposition 6.0.1.

Suppose $\tilde{\pi}$ is supercuspidal, choose $J' \subset J$ with $\#J' = m-1$, and set $\tilde{\pi}' = \tilde{\pi}_{[I\oplus J']}$. Then $\tilde{\pi}' \in \tilde{\Pi}_{\tilde{\phi}}(G^*_{b'})$, where $b' \in B(G^*)_{\mathrm{bas}}$ is the unique basic element with $\kappa_{G^*}(b')(-1) = (-1)^{m-1}$, and the induction hypothesis gives

$$\tilde{\phi}^{\mathrm{FS}}_{\tilde{\pi}'}|_{W_{K_1}} = \tilde{\phi}^{\mathrm{ss}}.$$

Corollary 3.4.2 shows that $\tilde{\pi}$ occurs in

$$\mathrm{Mant}_{G^*_{b'},b_1,\{\mu_1\}}(\tilde{\pi}_{[I\oplus J']}),$$

where $b_1 \in B(G^*_{b'})_{\mathrm{bas}}$ is the unique nontrivial basic element. Indeed, the error term has no supercuspidal constituents. Proposition 3.2.3 therefore gives

$$\tilde{\phi}^{\mathrm{FS}}_{\tilde{\pi}}|_{W_{K_1}} = \tilde{\phi}^{\mathrm{FS}}_{\tilde{\pi}_{[I\oplus J']}}|_{W_{K_1}} = \tilde{\phi}^{\mathrm{ss}}.$$

This completes the induction. □

7. Applications

7.1. **Unambiguous local Langlands correspondence for even orthogonal groups.** Combining the Fargues–Scholze correspondence with the classical correspondence, we obtain an unambiguous local Langlands correspondence for even special orthogonal groups: its parameters are defined up to conjugation by $\mathrm{SO}_{2n(G)}(\mathbb{C})$, rather than only by $\mathrm{O}_{2n(G)}(\mathbb{C})$. We retain the notation of §2. In particular, p is a rational prime, $K/\mathbb{Q}_p$ is unramified, and $(G = G^*_{b_0}, \varrho_{b_0}, z_{b_0})$ is a pure inner twist of G^* of type O2.

Theorem 7.1.1. *In Case O2, let $(G = G^*_{b_0}, \varrho_{b_0}, z_{b_0})$ be a pure inner twist of G^* such that $\mathrm{ord}_K(\mathrm{disc}(G)) \equiv 0 (\mathrm{mod}\ 2)$. Then there exists a map*

$$\mathrm{rec}^{\natural}_G : \Pi(G) \longrightarrow \Phi(G)$$

fitting into the following commutative diagram:

$$\begin{tikzcd} \Pi(G) \arrow[r, "\mathrm{rec}^{\natural}_G"] \arrow[d] & \Phi(G) \arrow[d] \\ \tilde{\Pi}(G) \arrow[r, "\mathrm{rec}_G"] & \tilde{\Phi}(G). \end{tikzcd}$$

For any $\phi \in \Phi(G)$, we write $\Pi_\phi(G) := (\mathrm{rec}^{\natural}_G)^{-1}(\phi)$, called the L-packet for ϕ. This correspondence satisfies the following properties:

(1) If $\phi \in \Phi(G)$ is not relevant for G in the sense of [KMSW14, Definition 0.4.14]*, then $\Pi_\phi(G) = \varnothing$.*

(2) For each $\phi \in \Phi(G)$ and $\pi \in \Pi_\phi(G)$, π is tempered if and only if ϕ is tempered, and π is a discrete series representation if and only if ϕ is discrete.

(3) $\mathrm{rec}^{\natural}_G$ only depends on G but not on ϱ_{b_0} and z_{b_0}. For the fixed Whittaker datum $\mathfrak{w}$ of G^*, there exists a canonical bijection

$$\iota_{\mathfrak{w},b_0} : \Pi_\phi(G) \xrightarrow{\sim} \mathrm{Irr}(\mathfrak{S}_\phi; \kappa_{b_0})$$

for each $\phi \in \Phi(G)$, where $\mathrm{Irr}(\mathfrak{S}_\phi; \kappa_{b_0})$ is the set of characters η of $\mathfrak{S}_\phi$ such that $\eta(z_\phi) = \kappa_{b_0}(-1)$. We write $\pi = \pi_{\mathfrak{w},b_0}(\phi, \eta)$ if $\pi \in \Pi_\phi(G)$ corresponds to $\eta \in \mathrm{Irr}(\mathfrak{S}_\phi; \kappa_{b_0})$ under $\iota_{\mathfrak{w},b_0}$.

(4) *(Compatibility with Langlands quotient) Let $P \leq G$ be a parabolic subgroup of G with a Levi factor*

$$M \cong \mathrm{Res}_{K_1/K}\mathrm{GL}_{d_1} \times \cdots \times \mathrm{Res}_{K_1/K}\mathrm{GL}_{d_r} \times G(n_0),$$

such that $M = \varrho_{b_0}(M^*)$ for a standard Levi subgroup M^* of G^*. Suppose $\pi \in \Pi(G)$ is the unique irreducible quotient of

$$\mathrm{I}_P^G\left((\tau_1 \otimes \nu^{s_1}) \boxtimes \cdots \boxtimes (\tau_r \otimes \nu^{s_r}) \boxtimes \pi_0\right),$$

where

$$d_1 + \cdots + d_r + n_0 = n(G), \qquad s_1 \geq \cdots \geq s_r > 0,$$

$\pi_0 \in \Pi_{\mathrm{temp}}(G(n_0))$ has parameter $\phi_0 := \mathrm{rec}^\natural_{G(n_0)}(\pi_0)$, and $\tau_i \in \Pi_{2,\mathrm{temp}}(\mathrm{GL}_{d_i})$ has parameter ϕ_i. Then $\mathrm{rec}^\natural_G(\pi)$ equals the image under ${}^LM \to {}^LG$ of

$$\left(\phi_1 \,|{-}|_{K_1}^{s_1}\right) \times \cdots \times \left(\phi_r \,|{-}|_{K_1}^{s_r}\right) \times \phi_0 \in \Phi(M).$$

Moreover, under the natural identification $\mathfrak{S}_{\phi_0} \cong \mathfrak{S}_\phi$, one has

$$\iota_{\mathfrak{w},b_0}(\pi) = \iota_{\mathfrak{w}_0,b_0}(\pi_0),$$

where $\mathfrak{w}_0$ is the induced Whittaker datum on M^*.

(5) *(Compatibility with standard γ-factors) Suppose $\pi \in \Pi(G)$ with $\phi := \mathrm{rec}^\natural_G(\pi)$, then for any character χ of $K^\times$,*

$$\gamma(\pi, \chi, \psi_K; s) = \gamma(\phi^{\mathrm{GL}} \otimes \chi, \psi_K; s),$$

where the left-hand side is the standard γ-factor defined by Lapid–Rallis using the doubling zeta integral [LR05] with the normalization modified in [GI14], and the right-hand side is the γ-factor defined in [Tat79].

(6) *(Compatibility with Plancherel measures) Suppose $\pi \in \Pi(G)$ with $\phi := \mathrm{rec}^\natural_G(\pi)$, then for any $d \in \mathbb{Z}_+$ and any $\tau \in \Pi(\mathrm{GL}_{d,K_1})$ with L-parameter ϕ_τ,*

$$\begin{aligned}\mu_{\psi_K}(\tau \otimes \nu^s \boxtimes \pi) =& \gamma(\phi_\tau \otimes (\phi^{\mathrm{GL}})^\vee, \psi_{K_1}; s) \cdot \gamma(\phi_\tau^\vee \otimes \phi^{\mathrm{GL}}, \psi_K^{-1}, -s) \\ &\times \gamma(\wedge^2(\phi_\tau), \psi_K; 2s) \cdot \gamma(\wedge^2(\phi_\tau^\vee), \psi_K^{-1}; -2s),\end{aligned}$$

where the left-hand side is the Plancherel measure defined in [GI14, §12]; cf. [GI16, §A.7].

(7) *(Local intertwining relations) Suppose $P \leq G$ is a maximal parabolic subgroup with a Levi factor*

$$M \cong \mathrm{GL}_d \times G(n-d),$$

$\tau \in \Pi_{2,\mathrm{temp}}(\mathrm{GL}_{d,K_1})$, $\pi_0 \in \Pi_{\mathrm{temp}}(G(n-d))$, $\phi_0 = \mathrm{rec}^\natural_{G(n-d)}(\pi_0)$, and $\pi \in \Pi_{\mathrm{temp}}(G)$ is a subrepresentation of $\mathrm{I}_P^G(\tau \boxtimes \pi_0)$. Assume that $M = \varrho_{b_0}(M^*)$ where M^* is a standard Levi subgroup of G^*, and $\mathfrak{w}_0$ is the induced Whittaker datum on M^*, then $\phi := \mathrm{rec}^\natural_G(\pi)$ equals the image of

$$\phi_\tau \times \phi_0 \in \Phi(M)$$

composed with the canonical embedding ${}^LM \to {}^LG$. Furthermore, if ϕ_τ is self-dual of sign 1 and the normalized intertwining operator

$$\mathrm{R}_{\mathfrak{w}}(w, \tau \boxtimes \pi_0) \in \mathrm{End}_{G(K)}\left(\mathrm{I}_P^G(\tau \boxtimes \pi_0)\right)$$

defined in [CZ21a, §7.1] acts on π by an element $\epsilon \in \{\pm 1\}$, where w is the unique nontrivial element in the relative Weyl group for M, then

$$\eta := \iota_{\mathfrak{w},b_0}(\pi) \in \mathrm{Irr}(\mathfrak{S}_\phi)$$

restricts to $\iota_{\mathfrak{w}_0,b_0}(\pi_0) \in \mathrm{Irr}(\mathfrak{S}_{\phi_0})$ under the natural embedding $\mathfrak{S}_{\phi_0} \hookrightarrow \mathfrak{S}_\phi$, and satisfies $\eta(e_\tau) = \epsilon$, where e_τ is the element of $\mathfrak{S}_\phi$ corresponding to ϕ_τ.

(8) *(Compatibility with Fargues–Scholze LLC) Suppose ℓ is a rational prime distinct from p with a fixed isomorphism $\iota_\ell : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}_\ell}$. For any $\pi \in \Pi(G)$ with L-parameter $\phi = \mathrm{rec}^\natural_G(\pi)$, the semisimplification satisfies $\phi^{\mathrm{ss}} = \iota_\ell^{-1}\phi^{\mathrm{FS}}_{\iota_\ell\pi} \in \Phi^{\mathrm{ss}}(G)$.*

Proof. We first explain why the Fargues–Scholze parameter singles out one of the two possible lifts of the classical parameter.

Let $\tilde{\phi} \in \tilde{\Phi}_2(G^*)$. By [GGP12, Theorem 8.1(ii)], the fiber of

$$\Phi_2(G^*) \longrightarrow \tilde{\Phi}_2(G^*)$$

over $\tilde{\phi}$ has cardinality two precisely when every irreducible summand

$$\rho \boxtimes \mathrm{sp}_a$$

of the standard representation $\tilde{\phi}^{\mathrm{GL}}$ has even dimension. We claim that, in this case, the same property remains true after semisimplification. Indeed, the semisimplification of such a summand is

$$(\rho \boxtimes \mathrm{sp}_a)^{\mathrm{ss}} = \left(\rho \otimes |-|_{K_1}^{\frac{a-1}{2}}\right) \oplus \left(\rho \otimes |-|_{K_1}^{\frac{a-3}{2}}\right) \oplus \cdots \oplus \left(\rho \otimes |-|_{K_1}^{\frac{1-a}{2}}\right).$$

Thus every irreducible summand of $(\rho \boxtimes \mathrm{sp}_a)^{\mathrm{ss}}$ has dimension $\dim(\rho)$. Suppose, for contradiction, that $\dim(\rho)$ is odd. Since

$$\dim(\rho \boxtimes \mathrm{sp}_a) = a \dim(\rho)$$

is even by assumption, a must be even. On the other hand, for an odd-dimensional conjugate self-dual representation ρ, its sign is $b(\rho) = 1$. Hence

$$b(\rho \boxtimes \mathrm{sp}_a) = b(\rho)(-1)^{a-1} = -1.$$

This contradicts the fact that $\rho\boxtimes\mathrm{sp}_a$ is a summand of the standard parameter for G^*, whose relevant summands have sign $b(G)$. Therefore $\dim(\rho)$ is even. Consequently every irreducible summand of $\tilde{\phi}^{\mathrm{GL,ss}}$ has even dimension.

Applying again [GGP12, Theorem 8.1(ii)], we see that the fiber of

$$\Phi^{\mathrm{ss}}(G^*) \longrightarrow \tilde{\Phi}^{\mathrm{ss}}(G^*)$$

over $\tilde{\phi}^{\mathrm{ss}}$ has the same cardinality as the fiber of

$$\Phi_2(G^*) \longrightarrow \tilde{\Phi}_2(G^*)$$

over $\tilde{\phi}$. In particular, the semisimplification map induces a bijection

$$\left\{\phi \in \Phi_2(G^*) \mid \phi \mapsto \tilde{\phi}\right\} \xrightarrow{\sim} \left\{\phi' \in \Phi^{\mathrm{ss}}(G^*) \mid \phi' \mapsto \tilde{\phi}^{\mathrm{ss}}\right\}.$$

This is the key point: the two possible lifts of $\tilde{\phi}$ remain distinct after semisimplification, so a semisimple Fargues–Scholze parameter can distinguish them.

Now let $\pi \in \Pi_{2,\mathrm{temp}}(G)$, and let

$$\tilde{\phi} = \mathrm{rec}_G(\pi) \in \tilde{\Phi}_2(G^*)$$

be its classical parameter. By Theorem A, the Fargues–Scholze parameter of π maps to the same ambiguous semisimple parameter:

$$\iota_\ell^{-1}\phi^{\mathrm{FS}}_{\iota_\ell\pi} \longmapsto \tilde{\phi}^{\mathrm{ss}} \qquad \text{in } \tilde{\Phi}^{\mathrm{ss}}(G^*).$$

Hence $\iota_\ell^{-1}\phi^{\mathrm{FS}}_{\iota_\ell\pi}$ belongs to the fiber over $\tilde{\phi}^{\mathrm{ss}}$. By the bijectivity just proved, there exists a unique element

$$\phi \in \Phi_2(G^*)$$

lying over $\tilde{\phi}$ such that

$$\phi^{\mathrm{ss}} = \iota_\ell^{-1}\phi^{\mathrm{FS}}_{\iota_\ell\pi}.$$

We define

$$\mathrm{rec}^{\natural}_G(\pi) := \phi.$$

This definition removes the outer-automorphism ambiguity in the classical correspondence: the classical parameter $\tilde{\phi}$ determines only an $\mathrm{O}_{2n}(\mathbb{C})$-conjugacy class, while the Fargues–Scholze parameter is a canonical semisimple $\mathrm{SO}_{2n}(\mathbb{C})$-conjugacy class. Since the two possible $\mathrm{SO}_{2n}(\mathbb{C})$-lifts have distinct semisimplifications, the Fargues–Scholze parameter selects exactly one of them.

This defines $\mathrm{rec}^{\natural}_G$ on discrete tempered representations. We extend it to all tempered representations by induction using the local intertwining relation in (7), exactly as in the proof of Proposition 2.4.3. More precisely, if a tempered representation occurs as a constituent of a normalized parabolic induction from a proper Levi subgroup, then the parameter on the Levi is already defined by the induction hypothesis, and (7) determines both the image parameter in $\Phi(G)$ and the internal parametrization of the packet. The preceding discrete tempered construction gives the initial case.

Finally, for a general irreducible smooth representation $\pi \in \Pi(G)$, we define $\mathrm{rec}^{\natural}_G(\pi)$ by the Langlands quotient property (4). Namely, write π as the unique irreducible quotient of the standard module

$$\mathrm{I}_P^G\left((\tau_1 \otimes \nu^{s_1}) \boxtimes \cdots \boxtimes (\tau_r \otimes \nu^{s_r}) \boxtimes \pi_0\right),$$

with $s_1 \geq \cdots \geq s_r > 0$, τ_i essentially square-integrable representations of the relevant general linear groups, and π_0 tempered. We then set $\mathrm{rec}^{\natural}_G(\pi)$ to be the image of

$$\left(\phi_1 \,|{-}|_{K_1}^{s_1}\right) \times \cdots \times \left(\phi_r \,|{-}|_{K_1}^{s_r}\right) \times \mathrm{rec}^{\natural}_{G(n_0)}(\pi_0)$$

under the canonical embedding ${}^LM \to {}^LG$.

The commutative diagram follows from the construction. Properties (1)–(3) are the corresponding properties of the classical correspondence, together with the definition of $\mathrm{rec}^{\natural}_G$ as the unique Fargues–Scholze-compatible lift. Properties (4) and (7) hold by construction. Properties (5) and (6) follow from the same properties for the classical correspondence in Theorem 2.3.1, since the standard representation ϕ^{GL} is unchanged by passing from the ambiguous parameter $\tilde{\phi}$ to its lift ϕ. Finally, property (8) is exactly the defining compatibility condition for discrete tempered representations, and for tempered and general representations it follows from the compatibility of the Fargues–Scholze correspondence with parabolic induction and Langlands quotients, Theorem 3.1.1. $\square$

Furthermore, the weak version of the Kottwitz conjecture Theorem 3.4.1 can be strengthened as follows:

Theorem 7.1.2. *Suppose $(G = G^*_{b_0}, \varrho_{b_0}, z_{b_0})$ is a pure inner twist of G^* associated with $b_0 \in B(G^*)_{\mathrm{bas}}$, μ is a dominant cocharacter of $G^*_{\overline{K}}$ and $b \in B(G^*, b_0, \mu)_{\mathrm{bas}}$ is the unique basic element. If $\phi \in \Phi_2(G^*)$ is a discrete L-parameter and $\rho \in \Pi_\phi(G_b)$, then*

$$\mathrm{Mant}_{G,b,\mu}(\iota_\ell \rho) = \sum_{\pi \in \Pi_\phi(G)} \dim \mathrm{Hom}_{\mathfrak{S}_\phi}(\delta[\pi,\rho], \mathcal{T}_\mu)[\iota_\ell \pi] + \mathrm{Err}$$

in $\mathrm{K}_0(G, \overline{\mathbb{Q}_\ell})$, where $\mathrm{Err} \in \mathrm{K}_0(G, \overline{\mathbb{Q}_\ell})$ is a virtual representation whose character vanishes on $G(K)_{\mathrm{s.reg,ell}}$.

Moreover, if ϕ is supercuspidal, then $\mathrm{Err} = 0$.

Proof. This follows from Theorem 3.4.1 and Proposition 3.2.3, by extracting the terms whose Fargues–Scholze parameters equal ϕ^{ss}, noting that $\mathrm{Mant}_{G,b,\{\mu\}}$ commutes with ς. $\square$

We conjecture that the correspondence defined in Theorem 7.1.1 satisfies the "unambiguous" endoscopic character identities of [Kal16]. This should reflect the expected compatibility of the

Fargues–Scholze correspondence with endoscopic transfer. More precisely, we expect that the endoscopic character identities follow from an ambiguous version of the conjectural averaging formula of Shin stated in [Ham24, Conjecture C.2], where ambiguity means we conflate representations conjugated by some outer automorphism. For example, in the trivial endoscopy case, using the Kottwitz conjecture Theorem 7.1.2, we prove the following endoscopic character identity between G and G^*.

Theorem 7.1.3. *Suppose $\phi \in \Phi_2(G^*)$, and $g \in G(K)_{\mathrm{s.reg,ell}}, h \in G^*(K)_{\mathrm{s.reg,ell}}$ are stably conjugate, then*

$$e(G) \sum_{\rho \in \Pi_\phi(G)} \Theta_\rho(g) = \mathrm{S}\Theta_\phi(h) = \sum_{\pi \in \Pi_\phi(G^*)} \Theta_\pi(h),$$

Proof. This is established by reversing the argument in the proof of Theorem 3.4.1. Given g, h, we may choose a dominant cocharacter μ of $G^*_{\overline{K}}$ such that $(g, h, \mu) \in \mathrm{Rel}_{b_0}$. Here we identify G_{b_0} with G^*. We apply Theorem 7.1.2, and it follows from Proposition 3.4.6 and the proof of Proposition 3.4.7 that for any $\pi \in \Pi_\phi(G^*)$,

$$e(G) \sum_{(h',\lambda)} \Theta_\pi(h') \dim \mathcal{T}_\mu[\lambda] = \sum_{\rho \in \Pi_\phi(G)} \dim \mathrm{Hom}_{\mathfrak{S}_\phi}(\delta[\rho, \pi], \mathcal{T}_\mu) \Theta_\rho(g). \tag{7.1}$$

For each $\rho \in \Pi_\phi(G)$, when π runs through all elements of $\Pi_\phi(G^*)$, $\delta[\rho, \pi]$ runs through every element of $\mathrm{Irr}(\mathfrak{S}_\phi; \kappa_{b_0})$ exactly once. Note that for the fixed g, a character $\lambda \in X_\bullet(Z_G(g)) = X^\bullet(\widehat{T})$ can be extended uniquely to a triple $(g, h', \lambda) \in \mathrm{Rel}_{b_0}$ if and only if the restriction of λ to $Z(\widehat{G})^{\mathrm{Gal}_K}$ equals κ_{b_0}; see [HKW22, p. 17]. So we sum Equation (7.1) over $\pi \in \Pi_\phi(G^*)$ to get

$$e(G) \sum_{(h',\lambda)} \dim \mathcal{T}_\mu[\lambda] \mathrm{S}\Theta_\phi(h') \tag{7.2}$$

$$= \dim \mathrm{Hom}_{Z(\widehat{G})^{\mathrm{Gal}_K}}(\kappa_{b_0}, \mathcal{T}_\mu) \sum_{\rho \in \Pi_\phi(G)} \Theta_\rho(g) \tag{7.3}$$

$$= \Big(\sum_{(h',\lambda)} \dim \mathcal{T}_\mu[\lambda] \Big) \cdot \sum_{\rho \in \Pi_\phi(G)} \Theta_\rho(g). \tag{7.4}$$

If there exists a pair (h', λ) such that $\mathcal{T}_\mu[\lambda] \neq 0$, $(g, h', \lambda) \in \mathrm{Rel}_{b_0}$ and $\lambda \neq \mu$, then we may replace μ by the dominant representative of λ to get a new equation. By the highest weight theory of representations, after finitely many steps we may replace the original μ by another μ' such that the only term in the summation of (7.2) is indexed by (h, μ'). Thus we get:

$$e(G) \sum_{\rho \in \Pi_\phi(G)} \Theta_\rho(g) = \mathrm{S}\Theta_\phi(h).$$

□

The arguments above show a general strategy to eliminate ambiguity in local Langlands correspondence caused by outer automorphisms: If we can construct a coarse local Langlands correspondence for all extended pure inner twists of a quasisplit reductive group G^* over K up to action of a finite group $\mathfrak{A}$ acting by outer automorphisms, character twists or taking contragredients, and verify endoscopic character identities in the sense of [Kal16] but up to action by $\mathfrak{A}$, then we may use it to deduce a weak version of the Kottwitz conjecture up to action by $\mathfrak{A}$. If we can also show the local Langlands correspondence constructed is compatible with Fargues–Scholze parameters up to action by $\mathfrak{A}$ in the sense of Theorem A, and the semisimplification map $\Phi(G) \to \Phi^{\mathrm{ss}}(G)$ is injective on each orbit of the $\mathfrak{A}$-action, then we may use the action of $\mathfrak{A}$ on the local shtuka space and [Kos21, Theorem 1.3] to extract a local Langlands correspondence not up to action by $\mathfrak{A}$. For example, we expect the strategy to hold for constructing local Langlands correspondence for pure inner forms of the even rank unitary similitude group $\mathrm{GU}(2n)$ with respect to unramified quadratic extensions or general even special orthogonal groups $\mathrm{GSO}(2n)$, following work of Xu [Xu16].

7.2. **Naturality of Fargues–Scholze LLC.** In this subsection, we will prove the following "naturality" property of Fargues–Scholze LLC for G, which is precisely [Ham24, Assumption 6.5], and a weaker result for a central extension of $\mathrm{Res}_{K/\mathbb{Q}_p}G$, which will be used to prove a vanishing result for relevant Shimura varieties in §8. We first recall some notation from [HL24, §4.2.1]: For a quasisplit reductive group G over a non-Archimedean local field K of characteristic zero with a Borel pair (B,T), if $\mathsf{b}\in B(\mathsf{G})_{\mathrm{un}}$, there exists a standard Levi subgroup $\mathsf{M_b}$ of G contained in a standard parabolic subgroup $\mathsf{P_b}$, such that $\mathsf{G_b}\cong\mathsf{M_b}$ under the inner twisting by b, and $\mathsf{B}\cap\mathsf{M_b}\leq\mathsf{M_b}$ transfers to a Borel subgroup $\mathsf{B_b}\leq\mathsf{G_b}$. Let $W_{\mathsf{b}}:=W_{\mathsf{G}}/W_{\mathsf{M_b}}$, and we identify any element of W_{b} with a representative in W_{G} of minimal length. For a character χ of $\mathsf{T}(K)$ and $\mathsf{w}\in W_{\mathsf{b}}$, we set

$$\rho^{\chi}_{\mathsf{b},\mathsf{w}}:=\mathrm{I}^{\mathsf{G_b}}_{\mathsf{B_b}}(\chi^{\mathsf{w}})\otimes\delta^{-1/2}_{\mathsf{P_b}}. \tag{7.5}$$

We now verify a property of the Fargues–Scholze LLC, which is stated as an assumption in [Ham24, Assumption 6.5].

Proposition 7.2.1.

(1) For each $b\in B(G^*)$, the classical LLC $\mathrm{rec}_{G^*_b}:\Pi(G^*_b)\to\Phi(G^*_b)$ is compatible with the Fargues–Scholze LLC, i.e., $\iota_\ell^{-1}\phi^{\mathrm{FS}}_{\iota_\ell\pi}=\phi^{\mathrm{ss}}_\pi$ for each $\pi\in\Pi(G^*_b)$.

(2) For $b\in B(G^*)$ and $\rho\in\Pi(G^*_b)$, let $\phi\in\Phi(G^*)$ be the composition of $\phi_\rho\in\Phi(G^*_b)$ with the twisted embedding

$$ {}^LG^*_b\xrightarrow{\sim}{}^LM_b\longrightarrow{}^LG^*,$$

as defined in [FS24, § IX.7.1], then ϕ factors through the canonical embedding ${}^LT^*\to{}^LG^*$ only if $b\in B(G^*)_{\mathrm{un}}$.

(3) In the situation of (2), if b is unramified and ϕ factors through $\phi_{T^*}\in\Phi(T^*)$, then ρ is isomorphic to an irreducible constituent of $\rho^\chi_{b,w}$, where $w\in W_b$ and χ is the character of T^* attached to ϕ_{T^*} via local Langlands correspondence for tori.

Proof. For (1): each G_b is an inner form of a Levi subgroup M_b of a parabolic subgroup of G^*, thus is of the form $\mathrm{Res}_{K_1/K}H\times G'$, where G' is a special orthogonal or unitary group that splits over an unramified quadratic extension of K, and H is a product of general linear groups. So the assertion follows from [HKW22, Theorem 6.6.1], Theorem A, and Theorem 7.1.1.

For (2): suppose $b\notin B(G^*)_{\mathrm{un}}$. By [Ham24, Lemma 2.12], G_b is not quasisplit. Consequently, T^* is not relevant for G_b in the sense of [KMSW14, Definition 0.4.14], and hence $\Pi_{\phi_\rho}(G_b)$ is empty. Indeed, by Jacquet–Langlands correspondence [DKV84], this follows from Theorem 7.1.1 together with the corresponding results for inner forms of general linear groups. This yields a contradiction.

For (3): first, we note ϕ is semisimple, since ${}^LT^*$ consists of semisimple elements. Since $G_b\cong M_b$, the preimage of ϕ under the natural embedding

$$\Phi(M^*_b)\longrightarrow\Phi(G^*)$$

is parametrized by a set of minimal length representatives of W_b. The assertion follows from the compatibility of the classical LLC with parabolic induction (Theorem 7.1.1). The factor $\delta^{-1/2}_{P^*_b}$ is included precisely to cancel the modulus-character twist appearing in the twisted L-embedding ${}^LG^*_b\cong{}^LM_b\to{}^LG^*$ defined in [FS24, §IX.7.1]. □

We also need to prove a weaker result for a central extension of $\mathrm{Res}_{K/\mathbb{Q}_p}G^*$. More generally, we impose the following global setup for future use:

Setup 7.2.2.

- Let F be a totally real number field and F_1 be either F or a CM field containing F, and let $\mathsf{c}\in\mathrm{Gal}(F_1/F)$ be the element with fixed field F,

- Let p be a rational prime that is unramified in F, with a fixed isomorphism $\iota_p : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}_p}$, and we write $K = F \otimes \mathbb{Q}_p$, which is a finite product of unramified finite extensions of $\mathbb{Q}_p$,
- Let $\mathbf{G}$ be a standard indefinite special orthogonal or unitary group over F defined by a c-Hermitian space $\mathbf{V}$ as in Definition 4.4.1, such that $G^* := \mathbf{G} \otimes_F K$ is quasisplit and splits over an unramified finite extension of $\mathbb{Q}_p$.
- for each quadratic imaginary element $\daleth \in \mathbb{R}_+\mathrm{i}$ (in particular $\daleth^2 \in \mathbb{Q}_-$), we have defined in §4.1 a central extension

$$1 \to \mathbf{Z}^{\mathbb{Q}} \to \mathbf{G}^\sharp \to \mathrm{Res}_{F/\mathbb{Q}}\mathbf{G} \to 1$$

where

$$\mathbf{Z}^{\mathbb{Q}} = \begin{cases} \{z \in \mathrm{Res}_{F(\daleth)/\mathbb{Q}}\mathrm{GL}_1 : \mathrm{Nm}_{F(\daleth)/F}(z) \in \mathbb{Q}^\times\} & \text{in Case O} \\ \{z \in \mathrm{Res}_{F_1/\mathbb{Q}}\mathrm{GL}_1 : \mathrm{Nm}_{F_1/F}(z) \in \mathbb{Q}^\times\} & \text{in Case U.} \end{cases}$$

Moreover, this central extension splits in Case U. In Case O, we assume that $\mathbb{Q}(\daleth)/\mathbb{Q}$ is split at p, so

$$\mathbf{Z}^{\mathbb{Q}} \otimes \mathbb{Q}_p \cong \mathrm{GL}_1 \times \mathrm{Res}_{K/\mathbb{Q}_p}\mathrm{GL}_1, \quad \mathbf{G}^\sharp \otimes \mathbb{Q}_p \cong \mathrm{GL}_1 \times \mathrm{Res}_{K/\mathbb{Q}_p} \mathrm{GSpin}(V^*).$$

Set

$$G^\sharp := \mathbf{G}^\sharp \otimes \mathbb{Q}_p, \tag{7.6}$$

which is a central extension of $\mathrm{Res}_{K/\mathbb{Q}_p}G^*$. We fix a Borel pair $(B^\sharp, T^\sharp)$ for $G^\sharp$ with image (B, T) in $\mathrm{Res}_{K/\mathbb{Q}_p}G^*$.

We now prove a version of Proposition 7.2.1 for those L-parameters of $G^\sharp$ coming from L-parameters of $\mathrm{Res}_{K/\mathbb{Q}_p}G^*$:

Theorem 7.2.3. *Let $\phi \in \Phi^{\mathrm{ss}}(\mathrm{Res}_{K/\mathbb{Q}_p}G^*)$ and let $\phi^\sharp$ be the image of ϕ under the natural L-homomorphism ${}^L(\mathrm{Res}_{K/\mathbb{Q}_p}G^*) \to {}^LG^\sharp$.*

(1) Suppose $b^\sharp \in B(G^\sharp)$ and $\rho^\sharp \in \Pi(G^\sharp_{b^\sharp})$. Assume that the image of $\phi^{\mathrm{FS}}_{\iota_\ell\rho^\sharp}$ under the twisted embedding

$${}^LG^\sharp_{b^\sharp} \cong {}^LM_{b^\sharp} \to {}^LG^\sharp,$$

as defined in [FS24, §IX.7.1], equals $\phi^\sharp$. Then $\rho^\sharp$ factors through a representation $\rho \in \Pi((\mathrm{Res}_{K/\mathbb{Q}_p}G^*)_b)$, where b is the image of $b^\sharp$ under the map $B(G^\sharp) \to B(\mathrm{Res}_{K/\mathbb{Q}_p}G^*)$, and the classical LLC $\phi_\rho \in \Phi((\mathrm{Res}_{K/\mathbb{Q}_p}G^*)_b)$ is defined, and the image of ϕ^{ss}_ρ under the natural L-homomorphism ${}^L(\mathrm{Res}_{K/\mathbb{Q}_p}G^*)_b \to {}^LG^\sharp_{b^\sharp}$ equals $\iota_\ell^{-1}\phi^{\mathrm{FS}}_{\iota_\ell\rho^\sharp}$.

(2) Situation as in (1), if $\phi^\sharp$ factors through the canonical embedding ${}^LT^\sharp \to {}^LG^\sharp$, then $b^\sharp \in B(G^\sharp)_{\mathrm{un}}$.

(3) In the situation of (1), if $b^\sharp$ is unramified and $\phi^\sharp$ factors through $\phi_{T^\sharp} \in \Phi^{\mathrm{ss}}(T^\sharp)$, then $\rho^\sharp$ is isomorphic to an irreducible constituent of $\rho^{\chi^\sharp}_{b^\sharp, w^\sharp}$, where $w^\sharp \in W_{b^\sharp}$ and $\chi^\sharp$ is the character of $T^\sharp$ attached to $\phi_{T^\sharp}$ via local Langlands correspondence for tori.

Proof. Note that $G^\sharp_{b^\sharp}$ is a product of Weil restrictions of general linear groups, unitary similitude groups or general spinor groups, so for any $b^\sharp \in B(G^\sharp)$ and $\rho^\sharp \in \Pi(G^\sharp_{b^\sharp})$ as in (1), $\rho^\sharp$ is trivial on the kernel Z of the map $G^\sharp_{b^\sharp}(\mathbb{Q}_p) \to (\mathrm{Res}_{K/\mathbb{Q}_p}G^*)_b(\mathbb{Q}_p)$, by compatibility of Fargues–Scholze LLC with central characters, Theorem 3.1.1. Thus $\rho^\sharp$ factors through a representation of $(\mathrm{Res}_{K/\mathbb{Q}_p}G^*)_b(\mathbb{Q}_p)$ because either the central extension

$$1 \to Z \to G^\sharp \to \mathrm{Res}_{K/\mathbb{Q}_p}G^* \to 1$$

is split or the kernel Z is an induced torus, i.e., a product of tori of the form $\mathrm{Res}_{L_i/\mathbb{Q}_p}\mathrm{GL}_1$ for finite extensions $L_i/\mathbb{Q}_p$. Now (1) follows from compatibility of Fargues–Scholze LLC with central extensions,4 Theorem 3.1.1, and (2)–(3) follow from Proposition 7.2.1 and the following Lemma 7.2.4 showing that $\Phi^{\mathrm{ss}}(\mathrm{Res}_{K/\mathbb{Q}_p}G^*) \to \Phi^{\mathrm{ss}}(G^\sharp)$ is injective. $\square$

Lemma 7.2.4. *Suppose K is a non-Archimedean local field of characteristic zero and $1 \to \mathsf{Z} \to \mathsf{G}' \to \mathsf{G} \to 1$ is a central extension of reductive groups over K such that Z is a torus and $\mathsf{G}'(K) \to \mathsf{G}(K)$ is surjective, then the natural homomorphism ${}^L\mathsf{G} \to {}^L\mathsf{G}'$ induces an injection $\Phi^{\mathrm{ss}}(\mathsf{G}) \hookrightarrow \Phi^{\mathrm{ss}}(\mathsf{G}')$.*

Proof. It follows from [Kot85, (1.9.1)] that there is a long exact sequence of pointed sets

$$1 \to \mathsf{Z}(K) \to \mathsf{G}'(K) \to \mathsf{G}(K) \to B(\mathsf{Z}) \to B(\mathsf{G}') \to B(\mathsf{G}).$$

So the hypothesis implies that $B(\mathsf{Z}) \to B(\mathsf{G}')$ is injective. By the Kottwitz isomorphism for tori, this implies that

$$Z(\widehat{\mathsf{G}'})^{\mathrm{Gal}_K} \to \widehat{\mathsf{Z}}^{\mathrm{Gal}_K}$$

is surjective, and hence so is

$$\widehat{\mathsf{G}'}^{\mathrm{Gal}_K} \to \widehat{\mathsf{Z}}^{\mathrm{Gal}_K}$$

Now apply the non-abelian cohomology exact sequence attached to $1 \to \widehat{\mathsf{G}} \to \widehat{\mathsf{G}'} \to \widehat{\mathsf{Z}} \to 1$. The preceding surjectivity implies that the induced map on continuous cocycles

$$\mathrm{H}^1(W_K, \widehat{\mathsf{G}}) \to \mathrm{H}^1(W_K, \widehat{\mathsf{G}'})$$

has trivial kernel. Thus $\Phi^{\mathrm{ss}}(\mathsf{G}) \to \Phi^{\mathrm{ss}}(\mathsf{G}')$ is injective. $\square$

7.3. **Strong Kottwitz conjecture.** We now revisit the Kottwitz conjecture discussed in §3.2 and §3.4. We combine the compatibility result and the Act-functors defined in §3.3 to describe the complex of $G(K) \times W_{E_\mu}$-modules $\mathrm{R}\Gamma_c(G, b, \mu)[\rho]$ without passing to the Grothendieck group and without the condition that μ is minuscule. We adopt the notation from §3, but without quotienting by outer automorphisms in Case O2, because we now have the unambiguous LLC Theorem 7.1.1. In particular, K is a non-Archimedean local field with residue characteristic p, and ℓ is a rational prime different from p with a fixed isomorphism $\iota_\ell : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}_\ell}$.

We first recall the following general result of Hansen [Han20, Theorem 1.1].

Theorem 7.3.1. *Suppose G is a quasisplit reductive group over $\mathbb{Q}_p$ with a Borel pair (B, T), $\mathsf{b}_0 \in B(\mathsf{G})_{\mathrm{bas}}$ is a basic element, μ is a minuscule dominant cocharacter of $\mathsf{G}_{\overline{\mathbb{Q}_p}}$, and $\mathsf{b} \in B(\mathsf{G}, \mathsf{b}_0, \mu^\bullet)$ (see (3.3)), so $(\mathsf{G}_{\mathsf{b}_0}, \mathsf{b}, \mu^\bullet)$ is a local Shimura datum in the sense of* [RV14, Definition 5.1], *and $\rho \in \Pi(\mathsf{G}_{\mathsf{b}}, \overline{\mathbb{Q}_\ell})$. Suppose the following conditions hold:*

(1) $\mathrm{Sht}(\mathsf{G}, \mathsf{b}, \mathsf{b}_0, \mu)$ *appears in the basic uniformization at p of a global Shimura variety in the sense of* Theorem 5.1.1*.*

(2) The Fargues–Scholze L-parameter ϕ_ρ^{FS} is supercuspidal.

Then the complex $\mathrm{R}\Gamma_c(\mathsf{G}, \mathsf{b}, \mathsf{b}_0, \mu^\bullet)[\rho]$ is concentrated in middle degree, which is 0 under our normalization.

Note that $\mathrm{Sht}(G^*, b_1, \mathbf{1}, \mu_1)$ appears in the basic uniformization of a global Shimura variety of Abelian type defined in §4.4, where $b_1 \in B(G^*)_{\mathrm{bas}}$ is the unique nontrivial basic element and μ_1 is defined in (2.5).

For the remainder of this subsection, fix a supercuspidal L-parameter $\phi \in \Phi_{\mathrm{sc}}(G^*)$ such that

$$\tilde{\phi}^{\mathrm{GL}} = \phi_1 \oplus \cdots \oplus \phi_k \oplus \phi_{k+1} \oplus \cdots \oplus \phi_r,$$

where the ϕ_i are distinct irreducible representations of W_{K_1} of dimension d_i, and d_i is odd if and only if $i \leq k$. We use the combinatorial notation on L-parameters introduced in §2.6. By the weak version of the Kottwitz conjecture together with the preceding theorem, the following holds.

Corollary 7.3.2. *For each subset $I \subset [r]_+$ with $\#I \equiv 1 (\mathrm{mod}\ 2)$, there is an isomorphism*

$$\mathrm{R\Gamma}_c(G^*, b_1, \mathbf{1}, \mu_1)[\iota_\ell \pi_{[I]}] \cong \bigoplus_{i=1}^{r} d_i \iota_\ell \pi_{[I \oplus \{i\}]}$$

of complexes of representations of $G^*(K)$.

Proof. It follows from Theorem A that $\iota_\ell^{-1} \phi^{\mathrm{FS}}_{\iota_\ell \pi_{[I]}} = \phi$ is supercuspidal, so Theorem 7.1.2 and Proposition 3.2.2 imply that

$$\left[\mathrm{R\Gamma}_c(G^*, b_1, \mathbf{1}, \mu_1)[\iota_\ell \pi_{[I]}]\right] = \sum_{i=1}^{r} d_i [\iota_\ell \pi_{[I \oplus \{i\}]}] \in \mathrm{K}_0(G^*, \overline{\mathbb{Q}_\ell}).$$

Moreover, Theorem 7.3.1 implies that $\mathrm{R\Gamma}_c(G^*, b_1, \mathbf{1}, \mu_1)[\iota_\ell \pi_{[I]}]$ is concentrated in degree 0. Hence it is a finite-length smooth representation of $G^*(K)$ that admits a finite filtration whose graded pieces consist of d_i copies of $\iota_\ell \pi_{[I \oplus \{i\}]}$ for each $i \in [r]_+$. By Corollary 2.5.2, $\pi_{[I \oplus \{i\}]} \in \Pi(G^*)$ is supercuspidal for each $i \in [r]_+$. Since supercuspidal representations are injective and projective in the category of smooth representations with a fixed central character, this filtration splits, and the assertion follows. $\square$

The Act-functors defined in §3.3 relate to the cohomology of local shtuka spaces via the following result of Fargues–Scholze [FS24, §X.2] and Hamann [Ham25, Corollary 3.11]. Recall that we write $\phi^\natural : W_K \to {}^L G$ for the homomorphism corresponding to the supercuspidal L-parameter $\phi \in \Phi^{\mathrm{sc}}(G)$; see §2.2.

Theorem 7.3.3. *Suppose that μ is a dominant cocharacter of $G^*_{\overline{K}}$ and $b \in B(G^*, b_0, \mu)_{\mathrm{bas}}$. The highest weight tilting module $\mathcal{T}_\mu$ of $\widehat{G}$ naturally extends to a representation $\mathcal{T}_\mu$ of $\widehat{G} \rtimes W_{E_\mu}$ as defined in* [Kot84, Lemma 2.1.2]*. Moreover, for each $\rho \in \Pi_\phi(G^*_b)$ there is an isomorphism*

$$\mathrm{R\Gamma}_c(G^*, b, b_0, \mu)[\iota_\ell \rho] \cong \bigoplus_{\eta \in \mathrm{Irr}(\mathfrak{S}_\phi)} \mathrm{Act}_\eta(\iota_\ell \rho) \boxtimes \iota_\ell \mathrm{Hom}_{\mathfrak{S}_\phi}\left(\eta, \mathcal{T}_\mu \circ \left(\phi^\natural|_{W_{E_\mu}}\right)\right)$$

of $G^*_{b_0}(K) \times W_{E_\mu}$-modules.

Combining this theorem with the monoidal property of Act-functors, we deduce the analogous result for non-minuscule μ, extending [Ham25, Theorem 8.2] and [MHN24, Theorem 4.6, Theorem 4.22] to special orthogonal groups and unitary groups.

Theorem 7.3.4. *For each $I \subset [r]_+$, there exists a bijection of multisets*

$$\{\mathrm{Act}_{[\{x\}]}(\pi_{[I]})\}_{x \in [r]_+} \cong \{\pi_{[I \oplus \{x\}]}\}_{x \in [r]_+}. \tag{7.7}$$

We choose a permutation $\sigma_\varnothing$ of $[r]_+$ such that

$$\mathrm{Act}_{[\{x\}]}(\pi_{[\varnothing]}) \cong \pi_{[\{\sigma_\varnothing(x)\}]}$$

for each $x \in [r]_+$. For each $b_0 \in B(G^*)_{\mathrm{bas}}$ and dominant cocharacter μ of $G^*_{\overline{K}}$ with reflex field E_μ, if $b \in B(G^*, b_0, \mu)_{\mathrm{bas}}$, we write

$$\mathcal{T}_\mu \circ \left(\phi^\natural|_{W_{E_\mu}}\right) = \bigoplus_{j=1}^{m_\mu} \eta_{[I_j^\mu]} \boxtimes \sigma_j^\mu$$

as a sum of irreducible representations of $\mathfrak{S}_\phi \times W_{E_\mu}$, where $I_j^\mu \in \mathscr{P}([r]_+)/\sim_k$ for $1 \leq j \leq m_\mu$. Then for any subset $I \subset [r]_+$ with $\#I \equiv \frac{\kappa_b(-1)-1}{2} (\mathrm{mod}\ 2)$, there is an isomorphism of $G^*_{b_0}(K) \times W_{E_\mu}$-modules

$$\iota_\ell^{-1} \mathrm{R\Gamma}_c(G^*, b, b_0, \mu)[\iota_\ell \pi_{[I]}] \cong \bigoplus_{j=1}^{m_\mu} \pi_{[I \oplus \sigma_\varnothing(I_j^\mu)]} \boxtimes \sigma_j^\mu. \tag{7.8}$$

Proof. If we apply Theorem 7.3.3 to $b = b_1, b_0 = \mathbf{1}$ and $\mu = \mu_1$, we get an isomorphism

$$\mathrm{R}\Gamma_c(G^*, b_1, \mathbf{1}, \mu_1)[\iota_\ell \pi_{[I]}] \cong \bigoplus_{j=1}^{r} \iota_\ell \mathrm{Act}_{[\{j\}]}(\pi_{[I]}) \boxtimes \phi_j$$

of representations of $G^*(K) \times W_{K_1}$ for each $I \subset [r]_+$ with $\#I \equiv 1 (\mathrm{mod}\ 2)$. So it follows from Corollary 7.3.2 and Schur's lemma (noting that each $\mathrm{Act}_{[j]}(\pi_{[I]})$ is irreducible) that there exists a bijection of multisets

$$\{\mathrm{Act}_{[\{x\}]}(\pi_{[I]})\}_{x \in [r]_+} \cong \{\pi_{[I \oplus \{x\}]}\}_{x \in [r]_+} \tag{7.9}$$

(The only subtlety comes from the case when $k = 2$, but it is easy to verify that this bijection holds in this case).

Next we consider $[I'] \in \mathscr{P}([r]_+)/\sim_k$ with $\#[I'] \equiv 0 (\mathrm{mod}\ 2)$. For each $j \in [r]_+$, it follows from the bijection (7.9) for $[I] = [I'] \oplus [\{j\}]$ that there exists $x_j \in [r]_+$ such that $\mathrm{Act}_{[\{x_j\}]}(\pi_{[I'] \oplus [\{j\}]}) \cong \pi_{[I']}$. Thus it follows from the monoidal property of Act recalled in §3.3 that $\mathrm{Act}_{[\{x_j\}]}(\pi_{[I']}) = \pi_{[I'] \oplus [\{j\}]}$. We then get a bijection of multisets

$$\{\mathrm{Act}_{[\{x_j\}]}(\pi_{[I']})\}_{j \in [r]_+} \cong \{\pi_{[I'] \oplus [\{j\}]}\}_{j \in [r]_+}. \tag{7.10}$$

Note that if $x_j = x_{j'}$, then $[I' \oplus \{j\}] = [I' \oplus \{j'\}]$. It follows from numerical counting that the left-hand side must equal the multiset $\{\mathrm{Act}_{[\{j\}]}(\pi_{[I']})\}_{j \in [r]_+}$, and there is a bijection of multisets

$$\{\mathrm{Act}_{[\{j\}]}(\pi_{[I']})\}_{j \in [r]_+} \cong \{\pi_{[I'] \oplus [\{j\}]}\}_{j \in [r]_+}. \tag{7.11}$$

Now the first assertion follows from (7.9) or (7.11) depending on the cardinality of I.

For the second assertion, by Theorem 7.3.3, it suffices to prove that

$$\mathrm{Act}_{[J]}(\pi_{[I]}) = \pi_{[I \oplus \sigma_\varnothing(J)]} \tag{7.12}$$

for all $I, J \subset [r]_+$. By the monoidal property of Act-functors recalled in §3.3, it suffices to prove

$$\mathrm{Act}_{[I]}(\pi_{[\varnothing]}) = \pi_{[\sigma_\varnothing(I)]} \tag{7.13}$$

for all $I \subset [r]_+$. We prove (7.13) by induction on $\#I$. The cases $\#I = 0$ and $\#I = 1$ follow respectively from the identity property of $\mathrm{Act}_{[\varnothing]}$ and from the definition of $\sigma_\varnothing$. Assume $\#I \geq 2$. For each $i \in I$, the induction hypothesis and monoidal property give

$$\mathrm{Act}_{[I]}(\pi_{[\varnothing]}) \cong \mathrm{Act}_{[\{i\}]}(\pi_{[\sigma_\varnothing(I \smallsetminus \{i\})]}). \tag{7.14}$$

By the first assertion already proved, there is a bijection of multisets

$$\{\mathrm{Act}_{[\{j\}]}(\pi_{[\sigma_\varnothing(I \smallsetminus \{i\})]})\}_{j \in [r]_+} \cong \{\pi_{[\sigma_\varnothing(I \smallsetminus \{i\})] \oplus [\{\sigma_\varnothing(j)\}]}\}_{j \in [r]_+},$$

For $j \in I \smallsetminus \{i\}$, the induction hypothesis gives

$$\mathrm{Act}_{[\{j\}]}(\pi_{[\sigma_\varnothing(I \smallsetminus \{i\})]}) \cong \pi_{[\sigma_\varnothing(I \smallsetminus \{i,j\})]} = \pi_{[\sigma_\varnothing(I \smallsetminus \{i\}) \oplus \{\sigma_\varnothing(j)\}]}.$$

Removing these common terms from the two multisets, we obtain

$$\{\mathrm{Act}_{[\{j\}]}(\pi_{[\sigma_\varnothing(I \smallsetminus \{i\})]})\}_{j \in [r]_+ \smallsetminus (I \smallsetminus \{i\})} \cong \{\pi_{[\sigma_\varnothing(I \smallsetminus \{i\}) \oplus \{\sigma_\varnothing(j)\}]}\}_{j \in [r]_+ \smallsetminus (I \smallsetminus \{i\})}. \tag{7.15}$$

For each $i \in I$, the object $\mathrm{Act}_{[I]}(\pi_{[\varnothing]})$ occurs in the left-hand multiset of (7.15), by (7.14). The intersection, over all $i \in I$, of the supports of the corresponding right-hand multisets consists only of $\pi_{[\sigma_\varnothing(I)]}$. Hence

$$\mathrm{Act}_{[I]}(\pi_{[\varnothing]}) \cong \pi_{[\sigma_\varnothing(I)]},$$

which proves (7.13), and therefore (7.12). ∎

In particular, understanding the part of the cohomology of local shtuka spaces with supercuspidal L-parameters is reduced to understanding the decomposition of $\mathcal{T}_\mu \circ \left(\phi^\natural|_{W_{E_\mu}}\right)$ as $\mathfrak{S}_\phi \times W_{E_\mu}$-modules, up to permutation of the L-packet $\Pi_\phi(G)$.

Theorem 7.3.4 implies that the strong Kottwitz conjecture [HKW22, Conjecture 1.0.1] is equivalent to the triviality of the permutation $\sigma_\varnothing$. For example, this is known for $G = \mathrm{U}(V)$ or $\mathrm{GU}(V)$ where V is an odd-dimensional Hermitian space with respect to the unramified quadratic extension $\mathbb{Q}_{p^2}/\mathbb{Q}_p$ by [BMN23].

Theorem 7.3.5. *If* Hypothesis 4.5.8 *holds, then $\sigma_\varnothing$ is trivial, and the strong Kottwitz conjecture holds.*

Proof. Fix $I \subset [r]_+$ with $\#I \equiv 1 (\mathrm{mod}\ 2)$. By Theorem 7.3.4, there is an isomorphism of $G^*(K) \times W_{K_1}$-modules

$$\iota_\ell^{-1}\mathrm{R}\Gamma_c(G^*, b_1, \mathbf{1}, \mu_1)[\iota_\ell \pi_{[I]}] \cong \bigoplus_{j=1}^{r} \pi_{[I \oplus \{\sigma_\varnothing(j)\}]} \boxtimes \phi_j. \tag{7.16}$$

Following §5.2, we globalize K_1/K to F_1/F and G^*/K to $\mathbf{G}/F$, and retain the notation from there. In particular, let $(\mathbf{V}, \mathbf{G} = \mathrm{U}(\mathbf{V})^\circ)$ be the standard indefinite pure inner form of $\mathbf{G}^*$ with $\mathbf{G}_\mathfrak{p} \cong G^*$, and let $(\mathbf{V}', \mathbf{G}' = \mathrm{U}(\mathbf{V}')^\circ)$ be the standard definite pure inner form of $\mathbf{G}^*$ with $\mathbf{G}'_\mathfrak{p} \cong G^*_{b_1}$.

Set $\rho := \pi_{[I]} \in \Pi(G^*_{b_1})$. We use the following globalization of ρ different from Proposition 5.2.1.

Lemma 7.3.6. *There exists*

- *a control tuple $\bigstar$ for $\mathbf{G}^*$ (see* Definition 4.3.1*) such that $\Sigma^{\mathrm{sc}} = \Sigma^\circ = \{\mathfrak{p}\}$ and $\Sigma^{\mathrm{St}} = \varnothing$, such that the highest weight of ξ is trivial except in Case O2, where it is*

$$(1, \ldots, 1)$$

  at every infinite place of F;[13]
- *a $\bigstar$-split compact open subgroup $\mathcal{K}^\mathfrak{p} \leq \mathbf{G}'(\mathbf{A}^\mathfrak{p}_{F,f})$ (see* Definition 4.3.3*);*
- *an automorphic representation $\Pi' = \otimes'_v \Pi'_v$ of $\mathbf{G}'(\mathbf{A}_F)$ such that*
  - *$\Pi'_\mathfrak{p} \cong \rho$ and $(\Pi'^{\mathfrak{p}}_f)^{\mathcal{K}^\mathfrak{p}} \neq 0$;*
  - *Π'_v is unramified for all $v \in \Sigma_F^{\mathrm{fin}} \setminus \Sigma$;*
  - *Π'_∞ is cohomological for ξ, i.e.,*

$$\mathrm{H}^i(\mathrm{Lie}(\mathbf{G}(F \otimes \mathbb{R})), \mathcal{K}_\infty, \Pi'_\infty \otimes_\mathbb{C} \xi) \neq 0$$

    for some $i \in \mathbb{N}$;
  - *The natural localization map*

$$\mathfrak{S}_{\boldsymbol{\psi}} \to \mathfrak{S}_\phi$$

    associated with the formal parameter $\boldsymbol{\psi}$ of Π' is an isomorphism.

Proof. Recall that

$$\tilde{\phi}^{\mathrm{GL}} = \phi_1 \oplus \cdots \oplus \phi_k \oplus \phi_{k+1} \oplus \cdots \oplus \phi_r.$$

We define global classical groups $\mathbf{G}_i$ over F for $1 \leq i \leq r$ as follows. In case U, let $\mathbf{G}_i$ denote the quasisplit global unitary group over F of geometric rank d_i that splits over F_1. In case O1, d_i is even for every $1 \leq i \leq r$, and we let $\mathbf{G}_i := \mathrm{SO}_{d_i+1}$. In case O2, we choose elements $\mathfrak{D}_i \in F^\times$ such that

- $(-1)^{d_i}\mathfrak{D}_i$ is totally positive for every $1 \leq i \leq r$.

[13] The representation ξ cannot be chosen to be trivial: otherwise, the formal parameter $\boldsymbol{\psi}$ of Π' would fail to be regular algebraic at the infinite places. For such parameters, local-global compatibility for the associated Galois representations is not yet known at ramified places.

- The extension $F_{\mathfrak{p}}(\sqrt{\mathfrak{D}_i})/F_{\mathfrak{p}}$ corresponds to $\det \circ \phi_i$ under the local Langlands correspondence for every $1 \leq i \leq r$, and
- $\mathrm{disc}(\mathbf{G}) \cdot \prod_{1 \leq i \leq r} \mathfrak{D}_i$ belongs to $(F^\times)^2$.

Then we set $\mathbf{G}_i = \mathrm{Sp}_{d_i - 1}$ for $1 \leq i \leq k$, and set $\mathbf{G}_i = \mathrm{SO}_{d_i}^{\mathfrak{D}_i}$ for $k+1 \leq i \leq r$.

Fix, for each $1 \leq i \leq r$, an irreducible representation π_i of $\mathbf{G}_i(F_{\mathfrak{p}})$ with L-parameter

$$\begin{cases} \phi_i & \text{in case U} \\ \phi_i \otimes (\det \circ \phi_i) & \text{in case O.} \end{cases}$$

Since $\mathbf{G}_i(F \otimes \mathbb{R})$ admits discrete series by Harish-Chandra's criterion and our choice of $\mathfrak{D}_i$, the standard Plancherel density theorem, for example [Shi12, Theorem 1.1.(i)], applies. In particular, if we choose an algebraic irreducible representation ξ_i of $\left(\mathrm{Res}_{F/\mathbb{Q}} \mathbf{G}_i\right) \otimes \mathbb{C}$, then it follows from [Shi12, Theorem 1.1.(i)] that there exists a ξ_i-cohomological cuspidal automorphic representation Π_i of $\mathbf{G}_i(\mathbf{A}_F)$ such that $\Pi_{i,\mathfrak{p}} \cong \pi_i$.

For each $1 \leq i \leq r$, Arthur's multiplicity formula, Theorem 4.2.1, implies that there exists an elliptic global A-parameter $\boldsymbol{\psi}_i$ for $\mathbf{G}_i$ such that Π_i belongs to the corresponding packet, i.e. $\Pi_i \in \mathcal{A}_{2,\boldsymbol{\psi}_i}(\mathbf{G}_i)$. Since the localization $\widetilde{\boldsymbol{\psi}}_{i,\mathfrak{p}}$ is isomorphic to $\phi_i \otimes (\det \circ \phi_i)$, $\boldsymbol{\psi}_i$ must be in fact an irreducible self-dual cuspidal automorphic representation of $\mathrm{GL}_{N(\mathbf{G}_i)}(\mathbf{A}_F)$. We then define an elliptic global A-parameter

$$\boldsymbol{\psi} = \begin{cases} \boldsymbol{\psi}_1 + \cdots + \boldsymbol{\psi}_r & \text{in case U} \\ (\boldsymbol{\psi}_1 \otimes \chi_{\mathfrak{D}_1}) + \cdots + (\boldsymbol{\psi}_r \otimes \chi_{\mathfrak{D}_r}) & \text{in case O} \end{cases}$$

for $\mathbf{G}$.

By choosing algebraic irreducible representations ξ_i of $\left(\mathrm{Res}_{F/\mathbb{Q}} \mathbf{G}_i\right) \otimes \mathbb{C}$ suitably, we may ensure that the restriction of the Archimedean L-parameter attached to the localization of the isobaric sum

$$\boldsymbol{\psi}_1 \boxplus \cdots \boxplus \boldsymbol{\psi}_r$$

at every infinite place to $W_{\mathbb{C}} \cong \mathbb{C}^\times$ is isomorphic to

$$\bigoplus_{i \in \mathfrak{I}} \arg^i$$

for

$$\mathfrak{I} := \begin{cases} \{N(\mathbf{G}), N(\mathbf{G}) - 2, \ldots, 2, -2, \ldots, -N(\mathbf{G})\} & \text{in case O2,} \\ \{N(\mathbf{G}) - 1, N(\mathbf{G}) - 3, \ldots, 3 - N(\mathbf{G}), 1 - N(\mathbf{G})\} & \text{otherwise.} \end{cases}$$

Finally, it follows from Arthur's multiplicity formula for $\mathbf{G}$ that there exists a ξ-cohomological automorphic representation $\Pi' = \otimes'_v \Pi'_v$ of $\mathbf{G}'(\mathbf{A}_F)$ with formal parameter $\boldsymbol{\psi}$, satisfying $\Pi'_{\mathfrak{p}} \cong \rho$. By choosing a sufficiently large finite subset of places Σ of F containing $\{\mathfrak{p}\}$ and Σ_F^∞, we may assume that Π'_v is unramified for all $v \in \Sigma_F^{\mathrm{fin}} \smallsetminus \Sigma$. We can also choose a $\bigstar$-split compact open subgroup $\mathscr{K}^{\mathfrak{p}} \leq \mathbf{G}'(\mathbf{A}_{F,f}^{\mathfrak{p}})$ such that $(\Pi_f'^{\mathfrak{p}})^{\mathscr{K}^{\mathfrak{p}}} \neq 0$. $\square$

Choose $\bigstar, \mathscr{K}^{\mathfrak{p}}$, and Π' as in Lemma 7.3.6. Let

$$\phi_{\Pi'}^{\Sigma} : \tilde{\mathbb{T}}^{\Sigma} \to \mathbb{C}$$

be the associated Hecke character of Π', and define $\mathfrak{m} := \iota_\ell \ker(\phi_{\Pi'}^{\Sigma})$, which is a maximal ideal of $\iota_\ell \tilde{\mathbb{T}}^{\Sigma}$. Following the argument of §5.2, we obtain an isomorphism

$$\begin{aligned} \Theta_{\mathfrak{m},\mathrm{sc}} : \mathrm{R}\Gamma_c(G^*, b_1, \mathbf{1}, \mu_1)_{\mathrm{sc}} \otimes \iota_\ell \left|-\right|_{K_1}^{\frac{-\dim_{\mathbb{C}}(\mathbf{X})}{2}} [\dim_{\mathbb{C}}(\mathbf{X})] \otimes^{\mathrm{L}}_{J(K)} \mathcal{A}(\mathbf{G}'(F) \backslash \mathbf{G}'(\mathbf{A}_{F,f}) / \mathscr{K}^{\mathfrak{p}}, \mathscr{L}_{\iota_\ell \xi})_{\mathfrak{m}} \\ \xrightarrow{\sim} \mathrm{R}\Gamma_c(\mathcal{S}_{\mathscr{K}^{\mathfrak{p}}}(\mathrm{Res}_{F/\mathbb{Q}} \mathbf{G}, \mathbf{X}), \mathcal{L}_{\iota_\ell \xi})_{\mathfrak{m}}. \end{aligned}$$

By further applying a Hecke projector, we may restrict to the terms corresponding to those automorphic representations $\dot{\Pi}'$ of $\mathbf{G}'(\mathbf{A}_F)$ cohomological for ξ such that $(\dot{\Pi}'_f)^{\Sigma}$ has associated Hecke character $\phi^{\Sigma}_{\Pi'}$. But each such $\dot{\Pi}'$ must satisfy $\tilde{\dot{\Pi}}'_{\mathfrak{p}} \cong \tilde{\Pi}'_{\mathfrak{p}}$, by our choice of Π' and Arthur's multiplicity formula.[14] By an analogue of Theorem 4.3.5, for each $1 \leq j \leq r$, there exists an automorphic representation Π_j of $\mathbf{G}(\mathbf{A}_F)$ satisfying that

- $\tilde{\Pi}_{j,v} \cong \tilde{\Pi}'_v$ via the isomorphism $\mathbf{G}(F_v) \cong \mathbf{G}'(F_v)$ induced by the inner twists for every $v \in \Sigma_F^{\mathrm{fin}} \smallsetminus \Sigma$, and
- $\tilde{\Pi}_{j,\mathfrak{p}} \cong \tilde{\pi}_{[I \oplus \{\sigma_{\varnothing}(j)\}]}$.

Combining with Equation (7.16), we see that ϕ_j is a submodule of the W_{K_1}-module $\rho_{\mathrm{Sh}}^{\Pi_j}$ for each $1 \leq j \leq r$. However, it follows from Arthur's multiplicity formula that $\sigma_{\varnothing}(j)$ is contained in $\mathfrak{J}(\Pi_j)$, and it follows from Hypothesis 4.5.8 that j is contained in $\mathfrak{J}(\Pi_j)$. Thus j and $\sigma_{\varnothing}(j)$ are both contained in $\mathfrak{J}(\Pi_j)$ for each $1 \leq j \leq r$. By the same hypothesis, either $\sigma_{\varnothing}(j) = j$ or $\mathfrak{J}(\Pi_j) = \{j, \sigma_{\varnothing}(j)\}$. In either case we have

$$\{j\} \oplus \{\sigma_{\varnothing}(j)\} \sim_k \varnothing.$$

As a result, $\sigma_{\varnothing}$ acts trivially on the quotient labelling, and we may replace it by the trivial permutation in Theorem 7.3.4. $\square$

Theorem 7.3.4 can be formulated more naturally in terms of eigensheaves. We recall from [Far16, Conjecture 4.4] that a *Hecke eigensheaf* for ϕ is an object $\mathcal{G}_\phi \in \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_{G^*}, \overline{\mathbb{Q}_\ell})$ such that, for any finite index set I and $(V, r_V) \in \mathrm{Rep}_{\overline{\mathbb{Q}_\ell}}({}^L G^{*I})$, there exists an isomorphism

$$\eta_{V,I} : \mathrm{T}_V(\mathcal{G}_\phi) \xrightarrow{\sim} \mathcal{G}_\phi \boxtimes r_V \circ \phi \in \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_{G^*}, \overline{\mathbb{Q}_\ell})^{BW_K^I} \tag{7.17}$$

that is natural in I and V and compatible with compositions of Hecke operators.

Following [MHN24, §4.1.3], we construct eigensheaves attached to the supercuspidal L-parameters ϕ:

Theorem 7.3.7. *Set*

$$\mathcal{G}_\phi := \bigoplus_{\eta \in \mathrm{Irr}(\mathfrak{S}_\phi)} \mathrm{Act}_\eta(\pi_{[\varnothing]}) \in \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_{G^*}, \overline{\mathbb{Q}_\ell}).$$

Let $\mathfrak{S}_\phi$ act on the summand $\mathrm{Act}_\eta(\pi_{[\varnothing]})$ through the character η.

- *$\mathcal{G}_\phi$ is supported on $B(G^*)_{\mathrm{bas}} \subset |\mathrm{Bun}_{G^*}|$, i.e., on the semi-stable locus of Bun_{G^*}.*
- *For each $b \in B(G^*)_{\mathrm{bas}}$, under the natural identification $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}^b_{G^*}, \overline{\mathbb{Q}_\ell})$ with $\mathrm{D}_{\mathrm{lis}}(G^*_b, \overline{\mathbb{Q}_\ell})$, there exists an isomorphism*

$$i_b^* \mathcal{G}_\phi \cong \bigoplus_{\substack{[I] \in \mathscr{P}([r]_+)/\sim_k, \\ \# I \equiv \frac{\kappa_b(-1)-1}{2} \pmod 2}} \eta_{[I]} \boxtimes \pi_{[\sigma_{\varnothing}(I)]}$$

of representations of $\mathfrak{S}_\phi \times G_b(K)$.
- *$\mathcal{G}_\phi$ is a Hecke eigensheaf for ϕ, i.e., (7.17) holds.*

Proof. These assertions follow from the bijection (7.7) and the symmetric monoidal property of the Act-functors recalled in §3.3, in the same way as in the proof of [MHN24, Proposition 4.18, Theorem 4.19]. $\square$

[14] Here we use that $\mathbf{G}'(F \otimes \mathbb{R})$ is compact, so each discrete series packet is a singleton.

8. A vanishing result for torsion cohomology of Shimura varieties

We use the compatibility result to prove a vanishing result for the generic part of the cohomology of orthogonal or unitary Shimura varieties with torsion coefficients.

We first recall the general torsion vanishing conjecture of [Han23, Ham24]. Let $(\mathbb{G}, \mathbb{X})$ be a Shimura datum with reflex field $E \subset \mathbb{C}$ (which is a number field), and let p be a rational prime coprime to $2 \cdot \#\pi_1([\mathbb{G}, \mathbb{G}])$, with a fixed isomorphism $\iota_p : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}_p}$. The isomorphism ι_p induces a place $\mathfrak{p}$ of E over p, and we write $\mathbb{C}_p$ for the completion of the algebraic closure of $E_{\mathfrak{p}} \subset \overline{\mathbb{Q}_p}$. We write $\mathsf{G} := \mathbb{G} \otimes \mathbb{Q}_p$. Let ℓ be a rational prime that is coprime to $p \cdot \#\pi_0(Z(\mathsf{G}))$, with a fixed isomorphism $\iota_\ell : \mathbb{C} \to \overline{\mathbb{Q}_\ell}$, which fixes a square root $\sqrt{p} \in \overline{\mathbb{Z}_\ell}$ thus also $\sqrt{p} \in \overline{\mathbb{F}}_\ell$. Let $\Lambda \in \{\overline{\mathbb{Q}_\ell}, \overline{\mathbb{F}_\ell}\}$. Whenever we consider $\overline{\mathbb{F}_\ell}$-coefficients, we assume that $\pi_0(Z(\mathsf{G}))$ is invertible in Λ to avoid complications in this ℓ-modular setting.

For neat compact open subgroup $\mathscr{K} \leq \mathbb{G}(\mathbf{A}_f)$, let $\mathcal{S}_{\mathscr{K}}(\mathbb{G}, \mathbb{X})$ be the adic space over $\mathrm{Spa}(E_{\mathfrak{p}})$ associated with the Shimura variety $\mathrm{Sh}_{\mathscr{K}}(\mathbb{G}, \mathbb{X})$. If $\mathscr{K}^p \leq \mathbb{G}(\mathbf{A}_f^p)$ is a neat compact open subgroup, we define

$$\mathcal{S}_{\mathscr{K}^p}(\mathbb{G}, \mathbb{X}) := \varprojlim_{\mathscr{K}_p} \mathcal{S}_{\mathscr{K}_p \mathscr{K}^p}(\mathbb{G}, \mathbb{X}),$$

where $\mathscr{K}_p$ runs through all open compact subgroups of $\mathbb{G}(\mathbb{Q}_p)$.

The $\mathsf{G}(\mathbb{Q}_p) \times W_{E_{\mathfrak{p}}}$-representation on

$$\mathrm{R}\Gamma_c(\mathcal{S}(\mathbb{G}, \mathbb{X})_{\mathscr{K}^p, \mathbb{C}_p}, \Lambda)$$

decomposes as

$$\mathrm{R}\Gamma_c(\mathcal{S}_{\mathscr{K}^p}(\mathbb{G}, \mathbb{X})_{\mathbb{C}_p}, \Lambda) = \bigoplus_{\phi} \mathrm{R}\Gamma_c(\mathcal{S}_{\mathscr{K}^p}(\mathbb{G}, \mathbb{X})_{\mathbb{C}_p}, \Lambda)_\phi$$

according to Fargues–Scholze parameters of irreducible subquotients, where ϕ runs through semisimple L-parameters $\phi \in \Phi^{\mathrm{ss}}(\mathsf{G}; \Lambda)$; see [HL24, Corollary 4.4].

We now recall the concept of (weakly) Langlands–Shahidi type L-parameters, as defined in [HL24, Definition 6.2].

Definition 8.0.1. Let G be a quasisplit reductive group over a non-Archimedean local field K of characteristic zero with a Borel pair (B, T). Let $\phi_{\mathsf{T}} \in \Phi^{\mathrm{ss}}(\mathsf{T}, \Lambda)$ be a semisimple L-parameter, and let

$$\phi \in \Phi^{\mathrm{ss}}(\mathsf{G}, \Lambda)$$

be its image under the natural embedding ${}^L\mathsf{T} \to {}^L\mathsf{G}$. We write $\phi_{\mathsf{T}}^\vee$ for the Chevalley dual of ϕ_{T}.

- we say ϕ is *generic* (or of Langlands–Shahidi type) if the following Galois cohomology complexes vanish

  $$\mathrm{R}\Gamma(W_K, {}^L\mathcal{T}_\mu \circ \phi_{\mathsf{T}}), \quad \mathrm{R}\Gamma(W_K, {}^L\mathcal{T}_\mu \circ \phi_{\mathsf{T}}^\vee)$$

  are both trivial for each positive coroot $\mu \in \Phi^\vee(\mathsf{G}, \mathsf{T})^+ \subset X_\bullet(\mathsf{T})$.[15]
- we say ϕ is of *weakly Langlands–Shahidi type* if the Galois cohomology groups

  $$\mathrm{H}^2(W_K, {}^L\mathcal{T}_\mu \circ \phi_{\mathsf{T}}), \quad \mathrm{H}^2(W_K, {}^L\mathcal{T}_\mu \circ \phi_{\mathsf{T}}^\vee)$$

  are both trivial for each positive coroot $\mu \in \Phi^\vee(\mathsf{G}, \mathsf{T})^+ \subset X_\bullet(\mathsf{T})$.

By [HL24, Remark 6.3], these conditions depend only on the image ϕ of ϕ_{T} under the natural embedding ${}^L\mathsf{T} \to {}^L\mathsf{G}$; we call ϕ a semisimple toral L-parameter for G. For a semisimple toral L-parameter ϕ, we also write $\phi^\vee$ for the image of $\phi_{\mathsf{T}}^\vee$ under the natural embedding ${}^L\mathsf{T} \to {}^L\mathsf{G}$.

Moreover, it follows from [HL24, Lemma 4.24] that, for a finite splitting field extension K'/K for G, ϕ is generic (resp. of weakly Langlands–Shahidi type) if and only if $\phi|_{W_{K'}}$ is. By local Tate duality, genericity for ϕ is equivalent to $\alpha \circ \phi|_{W_{K'}}$ not equaling $\mathbf{1}$ or $|-|_{K'}^{\pm 1}$ for each coroot α of G.

[15] Recall that ${}^LT_\mu \in \mathrm{Rep}_\Lambda({}^L\mathsf{T})$ is the extended highest weight tilting module attached to μ as defined in (3.6).

Going back to the global situation, we recall the following conjecture by Hamann and Lee [HL24, Conjecture 6.6] on the vanishing of cohomology of Shimura varieties with torsion coefficients:

Conjecture 8.0.2. Let $\phi \in \Phi^{\mathrm{ss}}(\mathsf{G}, \overline{\mathbb{F}_\ell})$ be a semisimple toral L-parameter of weakly Langlands–Shahidi type, then the complex $\mathrm{R}\Gamma_c(\mathcal{S}(\mathbb{G},\mathbb{X})_{\mathcal{K}^p,\mathbb{C}_p}, \overline{\mathbb{F}_\ell})_\phi$ (resp. $\mathrm{R}\Gamma(\mathcal{S}(\mathbb{G},\mathbb{X})_{\mathcal{K}^p,\mathbb{C}_p}, \overline{\mathbb{F}_\ell})_\phi$) is concentrated in degrees $0 \le i \le \dim_{\mathbb{C}}(\mathbb{X})$ (resp. $\dim_{\mathbb{C}}(\mathbb{X}) \le i \le 2\dim_{\mathbb{C}}(\mathbb{X})$).

Remark 8.0.3. Suppose $F^+ \neq \mathbb{Q}$ is a totally real field and $\mathbb{G} = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{U}(n,n)$ is the Weil restriction of a quasisplit unitary group of even rank, and we assume that $\mathbb{G}$ splits at p, i.e.

$$\mathbb{G} \otimes \mathbb{Q}_p \cong \prod_{i=1}^{[F^+:\mathbb{Q}]} \mathrm{GL}_{2n,\mathbb{Q}_p}.$$

Then the conjecture is true for any semisimple toral L-parameter $\phi \in \Phi^{\mathrm{ss}}(\mathbb{G}\otimes\mathbb{Q}_p, \overline{\mathbb{F}_\ell})$ of weakly Langlands–Shahidi type by [CS24, Theorem 1.1]. Note that if ϕ is an unramified character

$$\phi = \mathrm{diag}(\chi_1, \ldots, \chi_{2n}) \in \Phi(\mathrm{GL}_{2n,\mathbb{Q}_p}, \overline{\mathbb{F}_\ell}),$$

then ϕ is of weakly Langlands–Shahidi type if and only if $\chi_i \neq \chi_j \otimes |-|_{\mathbb{Q}_p}$ for any distinct $i, j \in [2n]_+$.

8.1. **Generic semisimple L-parameters.** In this subsection, we study generic semisimple toral L-parameters. We import Setup 7.2.2. In particular, F is a totally real number field unramified at a prime p, $\mathbf{G}$ is a special orthogonal or unitary group over F with $G^* = \mathbf{G} \otimes_F (F \otimes \mathbb{Q}_p)$, and $G^\sharp$ is a central extension of $\mathbf{G}$. We consider the Hodge cocharacter $\mu^\sharp$ of $\mathbf{G}^\sharp$ corresponding to the Deligne homomorphism $h_0^\sharp$ (see (4.3)). When viewed as a cocharacter of

$$(8.1)\quad G^\sharp_{\overline{\mathbb{Q}_p}} \cong \begin{cases} \mathrm{GL}_{1,\overline{\mathbb{Q}_p}} \times \left(\mathrm{Res}_{K/\mathbb{Q}_p}\mathrm{GSpin}(V^*)\right)_{\overline{\mathbb{Q}_p}} & = \mathrm{GL}_{1,\overline{\mathbb{Q}_p}} \times \prod_{v\in\mathrm{Hom}(K,\overline{\mathbb{Q}_p})} \mathrm{GSpin}(V^*)\otimes_{K,v}\overline{\mathbb{Q}_p} \\ & \quad\text{in Case O} \\ \mathrm{GL}_{1,\overline{\mathbb{Q}_p}} \times \left(\mathrm{Res}_{K/\mathbb{Q}_p}\mathrm{GU}(V^*)\right)_{\overline{\mathbb{Q}_p}} & = \mathrm{GL}_{1,\overline{\mathbb{Q}_p}} \times \prod_{v\in\mathrm{Hom}(K,\overline{\mathbb{Q}_p})} \mathrm{GU}(V^*)\otimes_{K,v}\overline{\mathbb{Q}_p} \\ & \quad\text{in Case U} \end{cases}$$

via the isomorphism ι_p, it is the inverse of the identity map on the GL_1-factor and nontrivial on exactly one other factor, where it is a lift $\mu_1^\sharp$ of the dominant inverse of the cocharacter μ_1 of $G^*_{\overline{K}} = G(V^*)^\circ_{\overline{K}}$ defined in (2.5).

We will prove a special case of Conjecture 8.0.2 for generic toral parameters ϕ with an additional condition called regularity; cf. [HL24, Definition 4.15].

Definition 8.1.1. Let G be a quasisplit reductive group over a p-adic field K with Borel pair (B,T). Let $\phi \in \Phi^{\mathrm{ss}}(\mathsf{G}, \overline{\mathbb{F}_\ell})$ be a generic toral semisimple L-parameter, and let χ be the character of $\mathsf{T}(K)$ attached to ϕ_{T} by the local Langlands correspondence for tori. We say that ϕ is *regular* if

$$\chi \ncong \chi^w$$

for every nontrivial element $w \in W_{\mathsf{G}}$.

We have the following descent property for regularity conditions.

Lemma 8.1.2. *Let G be a quasisplit reductive group over a p-adic field K with Borel pair (B,T), and let K'/K be a finite extension. If $\phi \in \Phi^{\mathrm{ss}}(\mathsf{G}, \overline{\mathbb{F}_\ell})$ is a semisimple toral L-parameter, then ϕ is regular, provided that $\phi|_{W_{K'}}$ has the same property.*

Proof. Let χ be the character of $\mathsf{T}(K)$ corresponding to ϕ_{T}. By local class field theory for tori, the character corresponding to $\phi_{\mathsf{T}}|_{W_{K'}}$ is

$$\chi \circ \mathrm{Nm}_{K'/K} : \mathsf{T}(K') \to \overline{\mathbb{F}_\ell}^\times,$$

where $\mathrm{Nm}_{K'/K} : \mathsf{T}(K') \to \mathsf{T}(K)$ denotes the norm map. This norm map is equivariant for the natural Weyl group action. Hence, if $\chi = \chi^w$ for some nontrivial $w \in W_{\mathsf{G}}$, then

$$\chi \circ \mathrm{Nm}_{K'/K} = (\chi \circ \mathrm{Nm}_{K'/K})^w,$$

so the restriction $\phi|_{W_{K'}}$ is not regular. This proves the lemma. □

For later use, we prove the following result regarding regularity of L-parameters and central extensions.

Lemma 8.1.3. *Suppose*

$$1 \to Z \to \mathsf{G}^\sharp \to \mathsf{G} \to 1$$

is a central extension of quasisplit reductive groups over a non-Archimedean local field K of characteristic zero, and assume that either this extension splits or Z is an induced torus. Let $(\mathsf{B}^\sharp, \mathsf{T}^\sharp)$ be a Borel pair of $\mathsf{G}^\sharp$ with image (B, T) in G. If $\phi \in \Phi^{\mathrm{ss}}(\mathsf{G}, \overline{\mathbb{F}_\ell})$ is a semisimple toral L-parameter which may be regarded as a semisimple toral L-parameter $\phi^\sharp \in \Phi^{\mathrm{ss}}(\mathsf{G}^\sharp, \overline{\mathbb{F}_\ell})$ via the canonical embedding

$${}^L\mathsf{G}(\overline{\mathbb{F}_\ell}) \to {}^L\mathsf{G}^\sharp(\overline{\mathbb{F}_\ell}),$$

then $\phi^\sharp$ is regular if and only if ϕ is.

Proof. The map $\mathsf{T}^\sharp(K) \to \mathsf{T}(K)$ is surjective: this is clear if the extension splits, and follows from Shapiro's lemma when Z is induced. Moreover, $W_{\mathsf{G}^\sharp} \cong W_{\mathsf{G}}$. The assertion follows immediately from the definition of regularity. □

We next discuss when genericity implies regularity. In Cases U and O1, genericity is sufficient for regularity. In Case O2 this is no longer true in general, because the $D_{n(G^*)}$-coroots do not include the characters $2\varepsilon_i$. We therefore impose the following additional genericity condition in Case O2.

Definition 8.1.4. Assume we are in Case O2. Let H be $\mathrm{Res}_{K/\mathbb{Q}_p} G^*$ or $G^\sharp$. We say that a semisimple toral L-parameter

$$\phi_{\mathsf{H}} \in \Phi^{\mathrm{ss}}(\mathsf{H}, \Lambda)$$

is $\widehat{\mathrm{Std}}$-*generic* if, after restricting to W_K and composing with the standard representation of any factor of the dual groups corresponding to some $v \in \mathrm{Hom}(K, \overline{\mathbb{Q}_p})$ (see (8.1)), the resulting semisimple toral parameter for $\mathrm{GL}_{2n(G^*)}$ is generic in the sense of Definition 8.0.1.

Equivalently, after base change to a finite splitting field K'/K and writing the noncentral toral part on a simple factor as

$$\phi_T = (\chi_1, \ldots, \chi_{n(G^*)}),$$

the characters occurring in $\widehat{\mathrm{Std}} \circ \phi_T$ are

$$\eta_1, \ldots, \eta_{2n(G^*)} = \chi_1, \ldots, \chi_{n(G^*)}, \chi_{n(G^*)}^{-1}, \ldots, \chi_1^{-1},$$

and for every pair of distinct entries η_a, η_b with $a \neq b$, one has

$$\eta_a \eta_b^{-1} \ncong \mathbf{1}, \qquad \eta_a \eta_b^{-1} \ncong |-|_{K'}^{\pm 1}.$$

This condition is stable under Chevalley duality.

The following result generalizes [HL24, Lemma 4.25] to special orthogonal groups and unitary groups.

Lemma 8.1.5. *If $\phi \in \Phi^{\mathrm{ss}}(\mathrm{Res}_{K/\mathbb{Q}_p} G^*, \overline{\mathbb{F}_\ell})$ is a generic semisimple toral L-parameter, and assume that ϕ is $\widehat{\mathrm{Std}}$-generic in case O2 (see* Definition 8.1.4*), then ϕ is regular. Furthermore, if we regard ϕ as a semisimple toral L-parameter $\phi^\sharp \in \Phi^{\mathrm{ss}}(G^\sharp, \overline{\mathbb{F}_\ell})$ via the natural embedding*

$${}^L(\mathrm{Res}_{K/\mathbb{Q}_p} G^*)(\overline{\mathbb{F}_\ell}) \to {}^L G^\sharp(\overline{\mathbb{F}_\ell}),$$

then it is also regular as a parameter for $G^\sharp(\overline{\mathbb{F}_\ell})$.

Proof. By Lemma 8.1.3, the second assertion follows from the first. For the first assertion, by Lemma 8.1.2 and [HL24, Lemma 4.24], it suffices to treat the split groups $\mathrm{SO}_{d(G^*),K}$ (in Case O) and the general linear group $\mathrm{GL}_{d(G^*),K_1}$ (in Case U). Under the isomorphism

$$\mathrm{GL}_1^{n(G^*)} \cong T^* : (t_1, \dots, t_{n(G^*)}) \mapsto \begin{cases} \mathrm{diag}(t_1, \dots, t_{n(G^*)}) & \text{in Case U,} \\ \mathrm{diag}(t_1, \dots, t_{n(G^*)}, 1, t_{n(G^*)}^{-1}, \dots, t_1^{-1}) & \text{in Case O1,} \end{cases}$$

W_{G^*} is $\mathrm{Sym}_{n(G^*)}$ in case U, and is the semi-direct product of the group $\mathrm{Sym}_{n(G^*)}$ acting by permutation on the group $\{\pm 1\}^{n(G^*)}$ in Case O1., and is the semi-direct product of the group $\mathrm{Sym}_{n(G^*)}$ acting by permutation on the kernel of the determinant map $\det : \{\pm 1\}^{n(G^*)} \to \{\pm 1\} : (\epsilon_1, \dots, \epsilon_{n(G^*)}) \mapsto \prod_i \epsilon_i$ in Case O2.

For each $i \in [n(G^*)]_+$, denote by ε_i the cocharacter $\mathrm{GL}_1 \to T^* : t \mapsto (1, \dots, t, 1, \dots)$ where the t is at the i-th coordinate; and set $\varepsilon_{-i} := -\varepsilon_i$.

We write $\phi = \chi_1 \boxtimes \cdots \boxtimes \chi_{n(G^*)}$, and write $\chi_{-i} := \chi_i^{-1}$ for $i \in [n(G^*)]$. Suppose, for contradiction, that

$$\chi = \chi^w \tag{8.2}$$

for some nontrivial element $w \in W_{G^*}$. Suppose ε_j is not fixed by W_{G^*}, and we write $\varepsilon_j^w = \varepsilon_k$, then we evaluate Equation (8.2) at ε_j to get

$$\chi_j = \chi_k.$$

In Case O1 or Case U, this contradicts genericity because $\varepsilon_j - \varepsilon_k$ is a coroot of $\mathrm{SO}_{d(G^*)}$ or $\mathrm{GL}_{d(G^*)}$. In Case O2, this contradicts genericity because the composition of $\varepsilon_j - \varepsilon_k$ with the standard representation of $\mathrm{SO}_{2n(G^*)}$ is a coroot of $\mathrm{GL}_{2n(G^*)}$. □

In the setting of Definition 8.1.1, assume that ϕ is generic and regular, then by [Ham24, Theorem 9.10] there exists an object

$$\mathrm{nEis}(\mathcal{S}_{\phi_{\mathsf{T}}}) \in \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_{\mathsf{G}}, \overline{\mathbb{F}_\ell}),$$

which is a perverse filtered Hecke eigensheaf on $\mathrm{Bun}_{\mathsf{G}}$ with eigenvalue ϕ in the sense of [Ham24, Theorem 6.1]. In particular, if μ is a dominant cocharacter of $\mathsf{G}_{\overline{K}}$ with extended highest weight tilting module ${}^L\mathcal{T}_\mu \in \mathrm{Rep}_{\overline{\mathbb{F}_\ell}}({}^L\mathsf{G})$ as defined in (3.6), and T_μ is the Hecke operator attached to ${}^L\mathcal{T}_\mu$ as defined in §3.1, then $\mathrm{T}_\mu(\mathrm{nEis}(\mathcal{S}_{\phi_{\mathsf{T}}}))$ admits a W_K-equivariant filtration indexed by Gal_K-orbits $\mathrm{Gal}_K .\nu_i$ in $X_\bullet(\mathsf{T})$. The graded piece indexed by $\mathrm{Gal}_K .\nu_i$ is

$$\mathrm{nEis}(\mathcal{S}_{\phi_{\mathsf{T}}}) \otimes ({}^L\mathcal{T}_{\nu_i} \circ \phi_{\mathsf{T}}) \otimes {}^L\mathcal{T}_\mu[\mathrm{Gal}_K .\nu_i].$$

Moreover, when this filtration splits, there is an isomorphism

$$\mathrm{T}_\mu(\mathrm{nEis}(\mathcal{S}_{\phi_{\mathsf{T}}})) \cong \mathcal{S}_{\phi_{\mathsf{T}}} \otimes {}^L\mathcal{T}_\mu \circ \phi$$

of sheaves in $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_{\mathsf{G}}, \overline{\mathbb{F}_\ell})^{\mathrm{B}W_K}$. We say that ϕ is *μ-regular* if this filtration splits. By [Ham24, Theorem 9.10], this μ-regularity condition is implied by the following strongly μ-regularity condition; cf. [HL24, Definition 4.16].

Definition 8.1.6. For a quasisplit reductive group G over a non-Archimedean local field K of characteristic zero with a Borel pair (B, T), and for a dominant cocharacter μ of $\mathsf{G}_{\overline{K}}$, a toral semisimple L-parameter $\phi \in \Phi^{\mathrm{ss}}(\mathsf{G}, \overline{\mathbb{F}_\ell})$ is called *strongly μ-regular* if

$$\mathrm{R}\Gamma(W_K, {}^L\mathcal{T}_{\nu-\nu'} \circ \phi_{\mathsf{T}})$$

is trivial for any weights ν, ν' of the extended highest weight tilting module ${}^L\mathcal{T}_\mu$ attached to μ that lie in distinct Gal_K-orbits.

For later use, we prove the following result regarding μ-regularity of L-parameters and central extensions:

Lemma 8.1.7. *Assume $\ell \neq 2$, and that $(\ell, n(G^*)!) = 1$ if we are in Case O1. Let*

$$\phi \in \Phi^{\mathrm{ss}}(G^*, \overline{\mathbb{F}_\ell})$$

be a generic semisimple toral L-parameter. In Case O2, assume in addition that ϕ is $\widehat{\mathrm{Std}}$-generic in the sense of Definition 8.1.4. *Then ϕ is μ-regular for every dominant cocharacter $\mu \in X_\bullet(G^*)$.*

Similarly, let

$$\phi^\sharp \in \Phi^{\mathrm{ss}}(G^\sharp, \overline{\mathbb{F}_\ell})$$

be a generic semisimple toral L-parameter. In Case O2, assume in addition that $\phi^\sharp$ is $\widehat{\mathrm{Std}}$-generic. Then $\phi^\sharp$ is $\mu^\sharp$-regular for every dominant cocharacter $\mu^\sharp \in X_\bullet(G^\sharp)$. In particular, $\mu^\sharp$ can be chosen to be not fixed by any nontrivial element of $W_{G^\sharp}$.

Proof. By base change [HL24, Lemma 4.24] and the isomorphism (8.1), it suffices to work after the relevant group is split. The assertion for general linear groups, i.e. Case U, is the argument of [HL24, Lemma 4.25]. We treat the orthogonal cases.

Let $\varepsilon_1, \dots, \varepsilon_{n(G^*)}$ be the standard coordinates of a split maximal torus. The weights of the standard representation $\widehat{\mathrm{Std}}$ of the dual group are

$$\pm\varepsilon_1, \dots, \pm\varepsilon_{n(G^*)}.$$

The standard representation extends to the standard representation of the corresponding dual similitude group, namely $\mathrm{GSp}_{2n(G^*)}$ in Case O1 and $\mathrm{GSO}_{2n(G^*)}$ in Case O2. We denote its highest weight by $\omega_1^\sharp$ on the similitude side.

In Case O1, the differences of two distinct weights of $\widehat{\mathrm{Std}}$ are of the form

$$\pm\varepsilon_i \pm \varepsilon_j \quad (i \neq j), \qquad \pm 2\varepsilon_i,$$

and these are coroots of G^*. Hence genericity implies strong ω_1-regularity for ϕ. The same argument applies to $\phi^\sharp$ for $\omega_1^\sharp$, since the central similitude character contributes the same scalar to every weight and therefore cancels in every weight difference.

In Case O2, the differences

$$\pm\varepsilon_i \pm \varepsilon_j \quad (i \neq j)$$

are coroots of G^* and are controlled by genericity. The remaining differences are

$$\pm 2\varepsilon_i,$$

which are not coroots of the $D_{n(G^*)}$-root system. These are exactly the additional ratios controlled by the $\widehat{\mathrm{Std}}$-genericity assumption. Thus ϕ is strongly ω_1-regular, and $\phi^\sharp$ is strongly $\omega_1^\sharp$-regular. Therefore they are ω_1-regular and $\omega_1^\sharp$-regular by [Ham24, Theorem 9.10].

It remains to pass from the standard representation to arbitrary dominant cocharacters. In Case O1, the highest weight tilting module with highest weight ω_i is realized as a direct summand of the appropriate exterior power $\wedge^i(\widehat{\mathrm{Std}})$ for $1 \le i \le n(G^*)$; the same statement holds for the corresponding highest weights $\omega_i^\sharp$ of the dual similitude group, up to a central character. This uses the assumption $(\ell, n!) = 1$; cf. [Jan03, pp. 286–287] and [Ham24, §9.1, Appendix B.2].

In Case O2, with $\overline{\mathbb{Q}_\ell}$-coefficients one has

$$\wedge^i(\widehat{\mathrm{Std}}) \cong \mathcal{T}_{\omega_i} \qquad 1 \le i \le n(G^*) - 2,$$
$$\wedge^{n(G^*)-1}(\widehat{\mathrm{Std}}) \cong \mathcal{T}_{\omega_{n(G^*)-1} + \omega_{n(G^*)}},$$

and

$$\wedge^{n(G^*)}(\widehat{\mathrm{Std}}) \cong \mathcal{T}_{2\omega_{n(G^*)-1}} \oplus \mathcal{T}_{2\omega_{n(G^*)}}.$$

These decompositions extend to the corresponding highest weights of the dual $\mathrm{GSO}_{2n(G^*)}$, again up to central characters. Since $\ell \neq 2$, the same direct-summand statements hold with $\overline{\mathbb{F}_\ell}$-coefficients; cf. [GW09, Theorem 5.5.13], [Jan03, pp. 286–287], and [Ham24, §9.1, Appendix B.2].

The noncentral parts of the dominant cocharacter monoids in the orthogonal cases are generated by the highest weights just listed. These highest weight tilting modules occur as direct summands of tensor products of $\widehat{\mathrm{Std}}$. Hence repeated application of [Ham24, Proposition 9.12] shows that ϕ is μ-regular for every dominant μ of G^*, and that $\phi^\sharp$ is $\mu^\sharp$-regular for every dominant $\mu^\sharp$ of $G^\sharp$. Central characters do not affect the condition, since they add a common character to all weights and hence disappear from weight differences.

Finally, we may choose $\tilde{\mu}^\sharp$ so that under the isomorphism (8.1)

$$G^\sharp_{\overline{\mathbb{Q}_p}} \cong \begin{cases} \mathrm{GL}_{1,\overline{\mathbb{Q}_p}} \times \prod_{v\in\mathrm{Hom}(K,\overline{\mathbb{Q}_p})} \mathrm{GSpin}(V^*) \otimes_{K,v} \overline{\mathbb{Q}_p} & \text{in Case O,} \\ \mathrm{GL}_{1,\overline{\mathbb{Q}_p}} \times \prod_{v\in\mathrm{Hom}(K,\overline{\mathbb{Q}_p})} \mathrm{GU}(V^*) \otimes_{K,v} \overline{\mathbb{Q}_p} & \text{in Case U} \end{cases}$$

it is of the form $(0, \tilde{\mu}^{\sharp\prime}, \ldots, \tilde{\mu}^{\sharp\prime})$, where $\tilde{\mu}^{\sharp\prime}$ is not fixed by any nontrivial Weyl group element. Then $\tilde{\mu}^\sharp$ is not fixed by any nontrivial element of $W_{G^\sharp}$. ◻

8.2. **Perverse t-exactness and vanishing results.** In this subsection, we prove a perverse t-exactness result for Hecke operators, and deduce a vanishing result for cohomology of Shimura varieties with torsion coefficients. We adopt the notation related to $\mathrm{Bun}_{\mathsf{G}}$ from §3.1.

For any reductive group G over a non-Archimedean local field K of characteristic zero and any open substack $U \subset \mathrm{Bun}_{\mathsf{G}}$, there exists a perverse t-structure on $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_{\mathsf{G}}, \overline{\mathbb{F}_\ell})$ defined as follows: For each $\mathsf{b} \in B(\mathsf{G})$, we define $d_{\mathsf{b}} := \langle 2\rho_{\mathsf{G}}, \nu_{\mathsf{b}}\rangle$, where ν_{b} is the slope homomorphism of b, cf. [HL24, Definition 4.14]. Then an object A is contained in ${}^p\mathrm{D}^{\leq 0}(U, \overline{\mathbb{F}_\ell})$ if $i_{\mathsf{b}}^* A \in \mathrm{D}^{\leq d_{\mathsf{b}}}(\mathsf{G}_{\mathsf{b}}, \Lambda)$, and A is contained in ${}^p\mathrm{D}^{\geq 0}(U, \overline{\mathbb{F}_\ell})$ if $i_{\mathsf{b}}^! A \in \mathrm{D}^{\geq d_{\mathsf{b}}}(\mathsf{G}_{\mathsf{b}}, \Lambda)$. Here we recall that i_{b} is the inclusion $\mathrm{Bun}_{\mathsf{G}}^{\mathsf{b}} \subset \mathrm{Bun}_{\mathsf{G}}$.

Let

$$\mathrm{D}^{\mathrm{ULA}}(\mathrm{Bun}_{\mathsf{G}}, \overline{\mathbb{F}_\ell}) \subset \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_{\mathsf{G}}, \overline{\mathbb{F}_\ell})$$

be the full subcategory of universally locally acyclic (ULA) objects; see [FS24, Definition IV.2.22]. By [FS24, Theorem V.7.1], $\mathrm{D}^{\mathrm{ULA}}(\mathrm{Bun}_{\mathsf{G}}, \overline{\mathbb{F}_\ell})$ consists of objects A such that $i_{\mathsf{b}}^* A \in \mathrm{D}^{\mathrm{adm}}(\mathsf{G}_{\mathsf{b}}, \overline{\mathbb{F}_\ell})$ for each $\mathsf{b} \in B(\mathsf{G})$.

We import Setup 7.2.2. In particular, F is a totally real number field unramified at a prime p, $\mathbf{G}$ is a special orthogonal or unitary group over F with $G^* = \mathbf{G} \otimes_F (F \otimes \mathbb{Q}_p)$, and $\mathsf{G}^\sharp$ is a central extension of $\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G}$ with $G^\sharp = \mathbf{G}^\sharp \otimes \mathbb{Q}_p$. We then have the following local result on the perverse t-exactness of Hecke operators, which generalizes [HL24, Corollary 4.27] to the present setting:

Theorem 8.2.1. *Suppose*

$$\phi \in \Phi^{\mathrm{ss}}(\mathrm{Res}_{K/\mathbb{Q}_p} G^*, \overline{\mathbb{F}_\ell})$$

is generic in the sense of Definition 8.0.1. *In Case O2, assume further that ϕ is $\widehat{\mathrm{Std}}$-generic in the sense of* Definition 8.1.4. *We regard ϕ as a semisimple toral L-parameter $\phi^\sharp \in \Phi^{\mathrm{ss}}(G^\sharp, \overline{\mathbb{F}_\ell})$ via the natural embedding*

$${}^L(\mathrm{Res}_{K/\mathbb{Q}_p} G^*)(\overline{\mathbb{F}_\ell}) \to {}^L G^\sharp(\overline{\mathbb{F}_\ell}).$$

Assume further that $\ell \neq 2$, and that $(\ell, n(G^*)!) = 1$ in Case O1. Then, for any dominant cocharacter $\mu^\sharp$ of $G^\sharp_{\overline{\mathbb{Q}_p}}$, the Hecke operator $\mathrm{T}_{\mu^\sharp}$ attached to the extended highest weight tilting module ${}^L\mathcal{T}_{\mu^\sharp}$, as defined in (3.2)*, preserves ULA objects. Moreover, the induced functor*

$$i_{\mathbf{1}}^* \mathrm{T}_{\mu^\sharp} : \mathrm{D}^{\mathrm{ULA}}(\mathrm{Bun}_{G^\sharp}, \overline{\mathbb{F}_\ell})_{\phi^\sharp} \to \mathrm{D}^{\mathrm{adm}}(G^\sharp, \overline{\mathbb{F}_\ell})_{\phi^\sharp},$$

where $\mathbf{1} \in B(G^\sharp)$ is the trivial element, is exact with respect to the perverse t-structure on the source and the natural t-structure on the target.

Proof. By [HL24, Theorem 4.23] and [Ham24, Theorem 1.13], it suffices to check that all of the following claims hold:

- [Ham24, Assumption 6.5] holds for $\phi^\sharp$.

- $\phi^\sharp$ is regular.
- $\phi^\sharp$ is $\mu^\sharp$-regular, and there exists a cocharacter $\tilde{\mu}^\sharp$ of $G^\sharp_{\overline{\mathbb{Q}_p}}$ that is not fixed by any nontrivial element $w \in W_{G^\sharp}$, such that $\phi^\sharp$ is $\tilde{\mu}^\sharp$-regular.
- $\rho^{\chi^\sharp}_{b^\sharp, w^\sharp}$, as defined in (7.5), is irreducible for any $b^\sharp \in B(G^\sharp)_{\mathrm{un}}$ and $w^\sharp \in W_{b^\sharp}$. Here the character $\chi^\sharp$ is attached to $\phi^\sharp_{\mathsf{T}^\sharp}$ via local Langlands correspondence for tori.

The first claim follows from Theorem 7.2.3. The regularity of $\phi^\sharp$ follows from Lemma 8.1.3 and Lemma 8.1.5. The third claim follows from Lemma 8.1.7.

It remains to prove the last claim. The group $G^\sharp_{b^\sharp}$ is isomorphic to a Levi factor of a parabolic subgroup of $G^\sharp$, which is of the form

$$\mathrm{GL}_{1,\mathbb{Q}_p} \times \mathrm{Res}_{K/\mathbb{Q}_p} G' \times \mathrm{Res}_{K_1/\mathbb{Q}_p} H,$$

where H is a product of general linear groups and G' is a general spinor or general unitary group over K that splits over an unramified quadratic extension. The preceding three claims hold with $G^\sharp$ replaced by $G^\sharp_{b^\sharp}$, and the desired irreducibility follows from [HL24, Lemma 4.21] and [Ham24, Proposition A.2]. $\square$

Next, we recall a perversity result, which will be a crucial ingredient in the proof of torsion vanishing result later. We impose the following global assumptions for future use:

Setup 8.2.2.

- $\mathrm{Sh}(\mathbb{G}, \mathbb{X})$ is proper, and there exists a Shimura datum of Hodge type $(\mathbb{G}^\sharp, \mathbb{X}^\sharp)$ with a map of Shimura data $(\mathbb{G}^\sharp, \mathbb{X}^\sharp) \to (\mathbb{G}, \mathbb{X})$ such that $\mathbb{G}^\sharp_{\mathrm{ad}} \to \mathbb{G}_{\mathrm{ad}}$ is an isomorphism and $\#\pi_0(Z(\mathbb{G}^\sharp))$ is coprime to ℓ. Let $E^\sharp \subset \mathbb{C}$ be the common reflex field.
- $\mathbb{G}$ and $\mathbb{G}^\sharp$ are unramified at p. Set

  $$\mathsf{G}^\sharp := \mathbb{G}^\sharp \otimes \mathbb{Q}_p, \qquad \mathsf{G} := \mathbb{G} \otimes \mathbb{Q}_p,$$

  and fix a Borel pair $(\mathsf{B}^\sharp, \mathsf{T}^\sharp)$ of $\mathsf{G}^\sharp$ with image (B, T) in G.
- The central extension $\mathsf{G}^\sharp \to \mathsf{G}$ extends to a map of reductive integral models $\mathcal{G}^\sharp \to \mathcal{G}$ over $\mathbb{Z}_p$, and we define

  $$\mathcal{K}^\sharp_p := \mathcal{G}^\sharp(\mathbb{Z}_p), \qquad \mathcal{K}_p := \mathcal{G}(\mathbb{Z}_p).$$
- $\mathcal{K}^{\sharp p} \le \mathbb{G}^\sharp(\mathbf{A}^p_f)$ is a neat compact open subgroup with image $\mathcal{K}^p$ in $\mathbb{G}(\mathbf{A}^p_f)$. Set

  $$\mathcal{K}^\sharp := \mathcal{K}^\sharp_p \mathcal{K}^{\sharp p} \le \mathbb{G}^\sharp(\mathbf{A}_f), \qquad \mathcal{K} := \mathcal{K}_p \mathcal{K}^p \le \mathbb{G}(\mathbf{A}_f).$$
- We write

  $$\mathcal{H}_{\mathcal{K}^\sharp_p} := \overline{\mathbb{F}_\ell}[\mathcal{K}^\sharp_p \backslash \mathsf{G}^\sharp(\mathbb{Q}_p)/\mathcal{K}^\sharp_p], \qquad \mathcal{H}_{\mathcal{K}_p} := \overline{\mathbb{F}_\ell}[\mathcal{K}_p \backslash \mathsf{G}(\mathbb{Q}_p)/\mathcal{K}_p]$$

  for the corresponding Hecke algebras with $\overline{\mathbb{F}_\ell}$-coefficients.
- Let $\mathfrak{m} \subset \mathcal{H}_{\mathcal{K}_p}$ be a maximal ideal with inverse image $\mathfrak{m}^\sharp \subset \mathcal{H}_{\mathcal{K}^\sharp_p}$ and corresponding semisimple toral L-parameters

  $$\phi_{\mathfrak{m}} \in \Phi^{\mathrm{ss}}(\mathsf{G}, \overline{\mathbb{F}_\ell}), \qquad \phi_{\mathfrak{m}^\sharp} \in \Phi^{\mathrm{ss}}(\mathsf{G}^\sharp, \overline{\mathbb{F}_\ell}),$$

  respectively.

We then have a finite Galois covering of Shimura varieties

$$\mathrm{Sh}_{\mathcal{K}^\sharp}(\mathbb{G}^\sharp, \mathbb{X}^\sharp) \to \mathrm{Sh}_{\mathcal{K}}(\mathbb{G}, \mathbb{X}).$$

over $E^\sharp$. Note that $\iota_p : \mathbb{C} \to \overline{\mathbb{Q}_p}$ induces an embedding $E^\sharp \to \overline{\mathbb{Q}_p}$.

We now use the Igusa stacks for Hodge type Shimura varieties defined in [DvHKZ24]. If we write

$$\mathrm{Sh}_{\mathcal{K}^{\sharp p}}(\mathbb{G}^\sharp, \mathbb{X}^\sharp) := \varprojlim_{\tilde{\mathcal{K}}_p} \mathrm{Sh}_{\tilde{\mathcal{K}}_p \mathcal{K}^{\sharp p}}(\mathbb{G}^\sharp, \mathbb{X}^\sharp),$$

where $\tilde{\mathcal{K}}_p$ runs through all compact open subgroups of $\mathsf{G}^\sharp(\mathbb{Q}_p)$, then we have the Hodge–Tate period map

$$\pi_{\mathrm{HT}} : \mathrm{Sh}_{\mathcal{K}^{\sharp p}}(\mathbb{G}^\sharp, \mathbb{X}^\sharp)^{\mathrm{an}} \to \mathrm{Gr}_{\mathsf{G}^\sharp, \mu^\sharp}.$$

We then have the following Igusa stack $\mathrm{Ig}_{\mathcal{K}^{\sharp p}}(\mathbb{G}^\sharp, \mathbb{X}^\sharp)$; see [DvHKZ24, Theorem I].

Theorem 8.2.3. *There is an Artin v-stack $\mathrm{Ig}_{\mathcal{K}^{\sharp p}}(\mathbb{G}^\sharp, \mathbb{X}^\sharp)$ on $\mathrm{Perfd}_{\overline{\kappa}}$ sitting in a Cartesian diagram*

$$\begin{tikzcd} \mathrm{Sh}_{\mathcal{K}^{\sharp p}}(\mathbb{G}^\sharp, \mathbb{X}^\sharp) \arrow[r, "\pi_{\mathrm{HT}}"] \arrow[d, "\mathrm{pr}_{\mathrm{Ig}}"] & \mathrm{Gr}_{\mathsf{G}^\sharp, \mu^\sharp} \arrow[d, "\mathrm{BL}"] \\ \mathrm{Ig}_{\mathcal{K}^{\sharp p}}(\mathbb{G}^\sharp, \mathbb{X}^\sharp) \arrow[r, "\pi_{\mathrm{HT}}^{\mathrm{Ig}}"] & \mathrm{Bun}_{\mathsf{G}^\sharp, \mu^{\sharp \bullet}}, \end{tikzcd}$$

where BL *is the Beauville–Laszlo map from* [FS24, Proposition III.3.1]. *Moreover, $\mathrm{Ig}_{\mathcal{K}^{\sharp p}}(\mathbb{G}^\sharp, \mathbb{X}^\sharp)$ is ℓ-cohomologically smooth of dimension 0, and its dualizing sheaf is isomorphic to $\overline{\mathbb{F}_\ell}[0]$.*

Define

$$\mathcal{F} := \mathrm{R}\pi^{\mathrm{Ig}}_{\mathrm{HT},*}(\overline{\mathbb{F}_\ell}) \in \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_{\mathsf{G}^\sharp, \mu^{\sharp\bullet}}, \overline{\mathbb{F}_\ell}),$$

which is universally locally acyclic by [DvHKZ24, Corollary 8.5.4]. Moreover, since $\mathrm{Sh}(\mathbb{G}^\sharp, \mathbb{X}^\sharp)$ is proper, we recall the following perversity result from [DvHKZ24, Theorem 8.6.3].

Theorem 8.2.4. *$\mathcal{F}$ is perverse, i.e.,*

$$\mathcal{F} \in {}^p\mathrm{D}^{\geq 0}(\mathrm{Bun}_{\mathsf{G}^\sharp, \mu^{\sharp\bullet}}, \overline{\mathbb{F}_\ell}) \cap {}^p\mathrm{D}^{\leq 0}(\mathrm{Bun}_{\mathsf{G}^\sharp, \mu^{\sharp\bullet}}, \overline{\mathbb{F}_\ell}).$$

The object $\mathcal{F}$ is related to cohomology of Shimura varieties as follows. By Theorem 8.2.3, [DvHKZ24, Theorem 8.4.10] yields the following relation between the cohomology of the relevant Shimura varieties and the value of the corresponding Hecke operator on $\mathcal{F}$.

Theorem 8.2.5. *There is an isomorphism*

$$\mathrm{R}\Gamma(\mathrm{Sh}_{\mathcal{K}^{\sharp p}}(\mathbb{G}^\sharp, \mathbb{X}^\sharp)_{\mathbb{C}_p}, \overline{\mathbb{F}_\ell}) \cong i_{\mathbf{1}}^* \mathrm{T}_{\mu^{\sharp\bullet}}(\mathcal{F}[-\dim_{\mathbb{C}}(\mathbb{X})]) \left(-\frac{\dim_{\mathbb{C}}(\mathbb{X})}{2}\right)$$

in $\mathrm{D}(\mathsf{G}^\sharp, \overline{\mathbb{F}_\ell})$.

We now state our first main theorem on the torsion vanishing of the cohomology of orthogonal or unitary Shimura varieties away from central dimension. We impose Setup 7.2.2 with $F \neq \mathbb{Q}$, and extend the map $G^\sharp \to \mathrm{Res}_{K/\mathbb{Q}_p} G$ to a map of reductive integral models $\mathcal{G}^\sharp \to \mathcal{G}$ over $\mathbb{Z}_p$ by fixing a $\mathbb{Z}_p$-lattice of $\mathrm{Res}_{K/\mathbb{Q}_p}(\mathbf{V} \otimes_F K)$, and define $\mathcal{K}_p^\sharp := \mathcal{G}^\sharp(\mathbb{Z}_p), \mathcal{K}_p := \mathcal{G}(\mathbb{Z}_p)$. Let $\mathcal{K}^{p,\sharp} \leq \mathbf{G}^\sharp(\mathbf{A}_f^p)$ be a compact open subgroup with image $\mathcal{K}^p \leq \left(\mathrm{Res}_{F/\mathbb{Q}} \mathbf{G}\right)(\mathbf{A}_f^p)$, such that

$$\mathcal{K} := \mathcal{K}_p \mathcal{K}^p \leq \mathrm{Res}_{F/\mathbb{Q}} \mathbf{G}(\mathbf{A}_f) \quad \text{and} \quad \mathcal{K}^\sharp := \mathcal{K}_p^\sharp \mathcal{K}^{\sharp p} \leq \mathbf{G}^\sharp(\mathbf{A}_f)$$

are both neat. Let ℓ be a rational prime coprime to p, and we assume moreover $(\ell, n(G)!) = 1$ in Case O1. Let

$$\mathcal{H}_{\mathcal{K}_p^\sharp} := \overline{\mathbb{F}_\ell}[\mathcal{K}_p^\sharp \backslash G^\sharp(\mathbb{Q}_p)/\mathcal{K}_p^\sharp], \qquad \mathcal{H}_{\mathcal{K}_p} := \overline{\mathbb{F}_\ell}[\mathcal{K}_p \backslash G^*(K)/\mathcal{K}_p]$$

be Hecke algebras with $\overline{\mathbb{F}_\ell}$-coefficients. Let $\mathfrak{m} \subset \mathcal{H}_{\mathcal{K}_p}$ be a maximal ideal with inverse image $\mathfrak{m}^\sharp \subset \mathcal{H}_{\mathcal{K}_p^\sharp}$ and corresponding semisimple toral L-parameters

$$\phi_{\mathfrak{m}} \in \Phi^{\mathrm{ss}}(\mathrm{Res}_{K/\mathbb{Q}_p} G, \overline{\mathbb{F}_\ell}), \quad \phi_{\mathfrak{m}^\sharp} \in \Phi^{\mathrm{ss}}(G^\sharp, \overline{\mathbb{F}_\ell}),$$

respectively. Moreover, we assume that p is coprime to $2\dim(\mathbf{V})$ in Case U, so we work in the setting of Setup 8.2.2.

Theorem 8.2.6. *Suppose $F \neq \mathbb{Q}$, and suppose*

$$\phi_{\mathfrak{m}} \in \Phi^{\mathrm{ss}}(\mathrm{Res}_{K/\mathbb{Q}_p} G, \overline{\mathbb{F}_\ell})$$

is generic. Assume further that $\phi_{\mathfrak{m}}$ is $\widehat{\mathrm{Std}}$-regular in the sense of Definition 8.1.4 *in case O2.*

(1) $\mathrm{H}^i_{\text{ét}}(\mathrm{Sh}_{\mathcal{K}^\sharp}(\mathbf{G}^\sharp, \mathbf{X}^\sharp)_{\overline{E^\sharp}}, \overline{\mathbb{F}_\ell})_{\mathfrak{m}^\sharp}$ *vanishes unless $i = \dim_{\mathbb{C}}(\mathbf{X})$.*

(2) $\mathrm{H}^i_{\text{ét}}(\mathrm{Sh}_{\mathcal{K}}(\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G}, \mathbf{X})_{\overline{E}}, \overline{\mathbb{F}_\ell})_{\mathfrak{m}}$ *vanishes unless $i = \dim_{\mathbb{C}}(\mathbf{X})$.*

Proof. The second assertion follows from the first one: $\mathrm{Sh}_{\mathcal{K}^\sharp}(\mathbf{G}^\sharp, \mathbf{X}^\sharp)_{\overline{E^\sharp}}$ is a finite Galois covering of an open and closed subset M of $\mathrm{Sh}_{\mathcal{K}}(\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G}, \mathbf{X})_{\overline{E^\sharp}}$ with Galois group denoted by $\mathfrak{T}$. The cohomology of M is equipped with Hecke action by $\mathcal{H}_{\mathcal{K}_p}$. By the Hochschild–Serre spectral sequence,

$$\mathrm{R}\Gamma\left(M, \overline{\mathbb{F}_\ell}\right)_{\mathfrak{m}} = \mathrm{R}\Gamma\left(\mathfrak{T}, \mathrm{R}\Gamma(\mathrm{Sh}_{\mathcal{K}^\sharp}(\mathbf{G}^\sharp, \mathbf{X}^\sharp)_{\overline{E^\sharp}}, \overline{\mathbb{F}_\ell})_{\mathfrak{m}^\sharp}\right)_{\mathfrak{m}},$$

which is concentrated in degree at least $\dim_{\mathbb{C}}(\mathbf{X})$ by (1). Now $\mathrm{R}\Gamma\left(\mathrm{Sh}_{\mathcal{K}}(\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G}, \mathbf{X})_{\overline{E}}, \overline{\mathbb{F}_\ell}\right)_{\mathfrak{m}}$ is a finite direct sum of copies of the complex $\mathrm{R}\Gamma\left(M, \overline{\mathbb{F}_\ell}\right)_{\mathfrak{m}}$, which is also concentrated in degree at least $\dim_{\mathbb{C}}(\mathbf{X})$.

But by the Poincaré duality and [HL24, Corollary A.7], it follows that

$$\mathrm{R}\Gamma(\mathrm{Sh}_{\mathcal{K}}(\mathbf{G}, \mathbf{X})_{\overline{E}}, \overline{\mathbb{F}_\ell})_{\mathfrak{m}} \cong \mathrm{R}\Gamma(\mathrm{Sh}_{\mathcal{K}}(\mathbf{G}, \mathbf{X})_{\overline{E}}, \overline{\mathbb{F}_\ell})_{\mathfrak{m}^\vee}(\dim_{\mathbb{C}}(\mathbf{X}))[2\dim_{\mathbb{C}}(\mathbf{X})]$$

is concentrated in degree at most $\dim_{\mathbb{C}}(\mathbf{X})$, where $\mathfrak{m}^\vee$ is the maximal ideal corresponding to $\phi_{\mathfrak{m}}^\vee$.

For the first assertion, note that $\mathrm{Sh}_{\mathcal{K}^\sharp}(\mathbf{G}^\sharp, \mathbf{X}^\sharp)$ is proper because $[\mathbf{G}^\sharp, \mathbf{G}^\sharp]$ is anisotropic, so the assertion follows from Theorems 8.2.4, 8.2.5, 8.2.1 and the Poincaré duality, as $\phi_{\mathfrak{m}}$ and $\phi_{\mathfrak{m}}^\vee$ are both generic. ∎

For Shimura varieties of Abelian type localized at a split place, the same argument gives a more general statement. In fact, we do not need the full strength of the compatibility of Fargues–Scholze LLC with "classical local Langlands correspondence" in the sense of [Ham24, Assumption 6.5], but only one property of the Fargues–Scholze LLC:

Hypothesis 8.2.7. Suppose G/K is a quasisplit reductive group with a Borel pair (B, T) and $\phi \in \Phi^{\mathrm{ss}}(\mathsf{G}, \overline{\mathbb{Q}_\ell})$ is a semisimple generic toral L-parameter. Then for any $\mathsf{b} \in B(\mathsf{G})$ and any $\rho \in \Pi(\mathsf{G}_{\mathsf{b}}, \overline{\mathbb{Q}_\ell})$, if the composition of $\phi_\rho^{\mathrm{FS}} : W_K \to {}^L\mathsf{G}_{\mathsf{b}}(\overline{\mathbb{Q}_\ell})$ with the twisted embedding ${}^L\mathsf{G}_{\mathsf{b}}(\overline{\mathbb{Q}_\ell}) \to {}^L\mathsf{G}(\overline{\mathbb{Q}_\ell})$ (as defined in [FS24, §IX.7.1]) equals ϕ, then b is unramified.

Remark 8.2.8. Note that, by [Ham24, Lemma 3.14], this hypothesis is a consequence of [Ham24, Assumption 6.5]: The hypothesis that $\phi^{\mathrm{ss}}_{\iota_\ell^{-1}\rho} = \iota_\ell^{-1}\phi_\rho^{\mathrm{FS}}$ factors through a generic parameter ϕ_{T} of T implies that $\phi_{\iota_\ell^{-1}\rho} = \phi^{\mathrm{ss}}_{\iota_\ell^{-1}\rho}$ by [Ham24, Lemma 3.14], which implies T is relevant for G_{b} by Theorem 2.3.1. So G_{b} is quasisplit, or equivalently, b is unramified.

Observe that this hypothesis is stable under duality: if it holds for ϕ, then it holds for $\phi^\vee$. The following invariance property may be of independent interest.

Proposition 8.2.9. *Suppose $\mathsf{G}' \to \mathsf{G}$ is a map of quasisplit reductive groups over a non-Archimedean local field K of characteristic zero that induces an isomorphism on adjoint groups. Let (B, T) and $(\mathsf{B}', \mathsf{T}')$ be compatible Borel pairs of G and G', respectively. If $\phi \in \Phi^{\mathrm{ss}}(\mathsf{G}, \overline{\mathbb{Q}_\ell})$ is a semisimple generic toral L-parameter, and we write $\phi' \in \Phi^{\mathrm{ss}}(\mathsf{G}', \overline{\mathbb{Q}_\ell})$ for the image of ϕ under the natural map $\Phi^{\mathrm{ss}}(\mathsf{G}, \overline{\mathbb{Q}_\ell}) \to \Phi^{\mathrm{ss}}(\mathsf{G}', \overline{\mathbb{Q}_\ell})$. Then* Hypothesis 8.2.7 *for ϕ implies that it holds for ϕ'.*

In particular, if $\mathsf{G}' \to \mathsf{G}$ is an injection, then Hypothesis 8.2.7 *for G implies that it holds for G'.*

Proof. Suppose $\mathsf{b}' \in B(\mathsf{G}')$ maps to $\mathsf{b} \in B(\mathsf{G})$, then there is a map $\mathsf{G}_{\mathsf{b}'} \to \mathsf{G}_{\mathsf{b}}$ that induces an isomorphism on adjoint groups. Suppose $\rho' \in \Pi(\mathsf{G}'_{\mathsf{b}'}, \overline{\mathbb{Q}_\ell})$ such that the composition of $\phi^{\mathrm{FS}}_{\rho'} : W_K \to {}^L\mathsf{G}'_{\mathsf{b}'}(\overline{\mathbb{Q}_\ell})$ with the twisted embedding ${}^L\mathsf{G}'_{\mathsf{b}'}(\overline{\mathbb{Q}_\ell}) \to {}^L\mathsf{G}'(\overline{\mathbb{Q}_\ell})$ equals $\phi' \in \Phi^{\mathrm{ss}}(\mathsf{G}', \overline{\mathbb{Q}_\ell})$. Then it follows from the compatibility of Fargues–Scholze parameters with central characters Theorem 3.1.1 that ρ' factors through $\mathrm{Im}(\mathsf{G}'_{\mathsf{b}'}(K) \to \mathsf{G}_{\mathsf{b}}(K))$. Then it follows from [GK82, Lemma 2.3] that there exists an irreducible smooth representation $\rho \in \Pi(\mathsf{G}_{\mathsf{b}}, \overline{\mathbb{Q}_\ell})$ such that ρ' is a subquotient of $\rho|_{\mathsf{G}'_{\mathsf{b}'}(K)}$. It follows from compatibility of Fargues–Scholze correspondence with central extensions, Theorem 3.1.1 that $\phi^{\mathrm{FS}}_{\rho'}$ is the image of $\phi^{\mathrm{FS}}_{\rho}$ under the natural map $\Phi^{\mathrm{ss}}(\mathsf{G}_{\mathsf{b}}, \overline{\mathbb{Q}_\ell}) \to \Phi^{\mathrm{ss}}(\mathsf{G}'_{\mathsf{b}'}, \overline{\mathbb{Q}_\ell})$. So the composition of $\phi^{\mathrm{FS}}_{\rho} : W_K \to {}^L\mathsf{G}_{\mathsf{b}}(\overline{\mathbb{Q}_\ell})$ with the twisted embedding ${}^L\mathsf{G}_{\mathsf{b}}(\overline{\mathbb{Q}_\ell}) \to {}^L\mathsf{G}(\overline{\mathbb{Q}_\ell})$ is a parameter $\tilde{\phi}$ whose image under the natural map $\Phi^{\mathrm{ss}}(\mathsf{G}, \overline{\mathbb{Q}_\ell}) \to \Phi^{\mathrm{ss}}(\mathsf{G}', \overline{\mathbb{Q}_\ell})$ equals ϕ'. So it follows from the compatibility of Fargues–Scholze correspondence with character twists and [Xu17, Appendix A] that we may twist ρ by a character of $\mathrm{Coker}(\mathsf{G}'_{\mathsf{b}'}(K) \to \mathsf{G}_{\mathsf{b}}(K))$ to make sure that $\tilde{\phi} = \phi$. Now the axiom for ϕ implies that b is unramified. In particular, G_{b} is quasisplit. Suppose $\mathsf{B}_{\mathsf{b}} \leq \mathsf{G}_{\mathsf{b}}$ is a Borel subgroup, then $\mathsf{B}_{\mathsf{b}'} = \mathsf{B}_{\mathsf{b}} \cap \mathsf{G}'_{\mathsf{b}'}$ is a Borel subgroup of $\mathsf{G}'_{\mathsf{b}'}$. Thus b' is unramified.

For the last assertion, it suffices to note that if $\phi \in \Phi^{\mathrm{ss}}(\mathsf{G}, \overline{\mathbb{Q}_\ell})$ has toral generic image $\phi' \in \Phi^{\mathrm{ss}}(\mathsf{G}', \overline{\mathbb{Q}_\ell})$, then ϕ is itself toral and generic. □

We now prove the second main theorem on vanishing result for torsion cohomology of Shimura varieties of Abelian type under Hypothesis 8.2.7. For a semisimple toral parameter $\overline{\phi} \in \Phi^{\mathrm{ss}}(\mathsf{G}, \overline{\mathbb{F}_\ell})$, we say that Hypothesis 8.2.7 holds for $\overline{\phi}$ if it holds for every semisimple generic toral lift $\phi \in \Phi^{\mathrm{ss}}(\mathsf{G}, \overline{\mathbb{Q}_\ell})$ whose reduction modulo ℓ is $\overline{\phi}$.

Theorem 8.2.10. *We work in the setting of* Setup 8.2.2*, and assume further that the set of unramified $\mu^\bullet_{\mathrm{ad}}$-acceptable elements $B(\mathsf{G}_{\mathrm{ad}}, \mu^\bullet_{\mathrm{ad}})_{\mathrm{un}}$ is a singleton. Suppose $\phi_{\mathfrak{m}}$ is generic and* Hypothesis 8.2.7 *holds for $\phi_{\mathfrak{m}}$ in the modulo ℓ sense.*

(1) $\mathrm{H}^i_{\text{ét}}(\mathrm{Sh}_{\mathcal{K}^\sharp}(\mathbb{G}^\sharp, \mathbb{X}^\sharp)_{\overline{E^\sharp}}, \overline{\mathbb{F}_\ell})_{\mathfrak{m}^\sharp}$ *vanishes unless $i = \dim_{\mathbb{C}}(\mathbb{X})$.*

(2) $\mathrm{H}^i_{\text{ét}}(\mathrm{Sh}_{\mathcal{K}}(\mathbb{G}, \mathbb{X})_{\overline{E^\sharp}}, \overline{\mathbb{F}_\ell})_{\mathfrak{m}}$ *vanishes unless $i = \dim_{\mathbb{C}}(\mathbb{X})$.*

Remark 8.2.11. By [XZ17, Corollary 4.2.4], the condition that $B(\mathsf{G}_{\mathrm{ad}}, \mu^\bullet_{\mathrm{ad}})_{\mathrm{un}}$ is a singleton (which is the μ_{ad}-ordinary element) is guaranteed when G_{ad} is a product of unramified Weil restrictions of split simple groups $\prod_{i=1}^k \mathrm{Res}_{L_i/\mathbb{Q}_p} \mathsf{H}_i$, and the conjugacy class of Hodge cocharacters $\{\mu\}$ associated with $\mathbb{X}^\sharp$ induces a dominant cocharacter $\mu_{\mathrm{ad}} = (\mu_1, \ldots, \mu_k)$ of $\mathsf{G}_{\overline{\mathbb{Q}_p}}$ via ι_p, such that each μ_i is trivial on all except possibly one simple factor of

$$(\mathrm{Res}_{L_i/\mathbb{Q}_p} \mathsf{H}_i)_{\overline{\mathbb{Q}_p}} \cong \prod_{v: L_i \hookrightarrow \overline{\mathbb{Q}_p}} \mathsf{H}_i \otimes_{L_i, v} \overline{\mathbb{Q}_p}.$$

Remark 8.2.12. The first assertion is established in [DvHKZ24, Theorem 10.1.6] under the assumption that the Fargues–Scholze LLC for $\mathsf{G}^\sharp$ is "natural" in the sense of [Ham24, Assumption 6.5]. However, this naturality is established in limited cases. For example, "classical local Langlands correspondence" for pure inner forms of GSpin_N when $N \geq 8$ has not been constructed, except for those irreducible representations with central character being a square of another character; see [GT19]. Nonetheless, we can still establish the theorem under this weaker axiom by modifying the argument in [DvHKZ24].

Proof. First, the second assertion follows from the first. Write $\mathcal{H}' = \mathrm{Im}(\mathcal{H}_{\mathcal{K}^\sharp_p} \to \mathcal{H}_{\mathcal{K}_p})$, and let $\mathfrak{m}'$ be the inverse image of $\mathfrak{m}$ in $\mathcal{H}'$. By functoriality of Shimura varieties, $\mathrm{Sh}_{\mathcal{K}^\sharp}(\mathbb{G}^\sharp, \mathbb{X}^\sharp)$ is a finite Galois covering of an open and closed subset M of $\mathrm{Sh}_{\mathcal{K}}(\mathbb{G}, \mathbb{X})_{\overline{E^\sharp}}$, with Galois group denoted by $\mathfrak{T}$. The cohomology of M is equipped with an action of $\mathcal{H}'$. By the Hochschild–Serre spectral sequence,

$$\mathrm{R}\Gamma\left(M, \overline{\mathbb{F}_\ell}\right)_{\mathfrak{m}'} = \mathrm{R}\Gamma\left(\mathfrak{T}, \mathrm{R}\Gamma(\mathrm{Sh}_{\mathcal{K}^\sharp}(\mathbb{G}^\sharp, \mathbb{X}^\sharp)_{\overline{E^\sharp}}, \overline{\mathbb{F}_\ell})_{\mathfrak{m}^\sharp}\right)_{\mathfrak{m}'}.$$

By (1), this complex is concentrated in degrees $\geq \dim_{\mathbb{C}}(\mathbb{X})$. Hence

$$\mathrm{RHom}_{\mathcal{H}'_{\mathfrak{m}'}}\left((\mathcal{H}_{\mathcal{K}_p})_{\mathfrak{m}}, \mathrm{R}\Gamma\left(M, \overline{\mathbb{F}_\ell}\right)_{\mathfrak{m}'}\right)$$

is concentrated in degrees $\geq \dim_{\mathbb{C}}(\mathbb{X})$, and so is

$$\mathrm{R}\Gamma\left(\mathrm{Sh}_{\mathcal{K}}(\mathbb{G}, \mathbb{X})_{\overline{E^\sharp}}, \overline{\mathbb{F}_\ell}\right)_{\mathfrak{m}}.$$

Applying the same argument to the dual maximal ideal $\mathfrak{m}^\vee$ corresponding to $\phi_{\mathfrak{m}}^\vee$, and using Poincaré duality together with [HL24, Corollary A.7], we obtain

$$\mathrm{R}\Gamma(\mathrm{Sh}_{\mathcal{K}}(\mathbb{G}, \mathbb{X})_{\overline{E^\sharp}}, \overline{\mathbb{F}_\ell})_{\mathfrak{m}} \cong \mathrm{R}\Gamma(\mathrm{Sh}_{\mathcal{K}}(\mathbb{G}, \mathbb{X})_{\overline{E^\sharp}}, \overline{\mathbb{F}_\ell})_{\mathfrak{m}^\vee}(\dim_{\mathbb{C}}(\mathbb{X}))[2\dim_{\mathbb{C}}(\mathbb{X})].$$

Thus the $\mathfrak{m}$-localized cohomology is also concentrated in degrees $\leq \dim_{\mathbb{C}}(\mathbb{X})$, proving (2).

For the first assertion, note that $\phi_{\mathfrak{m}^\sharp}$ and $\phi_{\mathfrak{m}^\sharp}^\vee$ are both generic. By Theorem 8.2.4 and 8.2.5, together with Poincaré duality, it suffices to show the t-exactness of

$$i_{\mathbf{1}}^* \mathrm{T}_{\mu^{\sharp\bullet}} : \mathrm{D}^{\mathrm{ULA}}(\mathrm{Bun}_{\mathsf{G}^\sharp}, \overline{\mathbb{F}_\ell})_{\overline{\phi}^\sharp} \to \mathrm{D}^{\mathrm{adm}}(\mathsf{G}^\sharp, \overline{\mathbb{F}_\ell})_{\overline{\phi}^\sharp}$$

(as defined in Theorem 8.2.1) for any semisimple generic toral L-parameter $\overline{\phi} \in \Phi^{\mathrm{ss}}(\mathsf{G}, \overline{\mathbb{F}_\ell})$ satisfying Hypothesis 8.2.7 in the modulo ℓ sense, where $\overline{\phi}^\sharp$ denotes its image under

$${}^L\mathsf{G}(\overline{\mathbb{F}_\ell}) \to {}^L\mathsf{G}^\sharp(\overline{\mathbb{F}_\ell}).$$

By [XZ17, Corollary 4.2.4], the natural image of $B(\mathsf{G}^\sharp, \mu^{\sharp\bullet})_{\mathrm{un}}$ in $B(\mathsf{G}_{\mathrm{ad}}, \mu_{\mathrm{ad}}^\bullet)_{\mathrm{un}}$ is a singleton by assumption. Thus, by [DvHKZ24, Proposition 10.2.4], it remains to show that $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_{\mathsf{G}^\sharp}, \overline{\mathbb{F}_\ell})_{\overline{\phi}^\sharp}$ is supported on the unramified strata.

Let

$$\overline{\rho}^\sharp \in \mathrm{D}(\mathrm{Bun}_{\mathsf{G}^\sharp}^{\mathsf{b}^\sharp}, \overline{\mathbb{F}_\ell})_{\overline{\phi}^\sharp} \cong \mathrm{D}(\mathsf{G}_{\mathsf{b}^\sharp}^\sharp, \overline{\mathbb{F}_\ell})$$

be an irreducible admissible representation of $\mathsf{G}_{\mathsf{b}^\sharp}^\sharp(\mathbb{Q}_p)$. By [HL24, Lemma 4.3(1)] and [FS24, §IX.7.1], the Fargues–Scholze parameter of $\overline{\rho}^\sharp$, composed with the twisted embedding

$${}^L\mathsf{G}_{\mathsf{b}^\sharp}^\sharp(\overline{\mathbb{F}_\ell}) \to {}^L\mathsf{G}^\sharp(\overline{\mathbb{F}_\ell}),$$

is $\overline{\phi}^\sharp$.

By [Dat05, Lemma 6.8], lift $\overline{\rho}^\sharp$ to an irreducible admissible $\overline{\mathbb{Q}_\ell}$-representation $\rho^\sharp$ admitting a $\mathsf{G}_{\mathsf{b}^\sharp}^\sharp(\mathbb{Q}_p)$-stable $\overline{\mathbb{Z}_\ell}$-lattice such that $\overline{\rho}^\sharp$ occurs as a subquotient of its reduction modulo ℓ. By compatibility of Fargues–Scholze parameters with reduction modulo ℓ, Theorem 3.1.1, the parameter $\phi_{\rho^\sharp}^{\mathrm{FS}}$ has reduction equal to $\phi_{\overline{\rho}^\sharp}^{\mathrm{FS}}$. Hence its composition with the twisted embedding

$${}^L\mathsf{G}_{\mathsf{b}^\sharp}^\sharp(\overline{\mathbb{Q}_\ell}) \to {}^L\mathsf{G}^\sharp(\overline{\mathbb{Q}_\ell})$$

is a semisimple generic toral lift of $\overline{\phi}^\sharp$. By the modulo ℓ form of Hypothesis 8.2.7 and Proposition 8.2.9, the element $\mathsf{b}^\sharp$ is unramified. This proves the required support statement. □

Appendix A. (Twisted) Endoscopy theory

In this appendix we review some standard definitions related to the trace formulas used in the main text and fix notation. §A.1 recalls (twisted) endoscopic triples for reductive groups over both local and global fields. §A.2 reviews local transfers and pseudo-coefficients for square-integrable irreducible admissible representations. §A.4 records Lefschetz functions and also a simple stable trace formula that will be used in the Langlands–Kottwitz method §4.5.

A.1. **Endoscopic triples.** We recall here some general definitions of extended endoscopic triples from [LS87, §1.2] and [KS99, §2.1]. Note that the more general notion of endoscopic data is not needed in the cases we consider in this paper.

Let F be a local or global field of characteristic zero, and consider a pair (G^*, θ^*) where

(1) G^* is a quasisplit reductive group over F with a fixed Gal_F-invariant pinning on $\widehat{\mathsf{G}^*}$,

(2) θ^* is a pinned automorphism of G^*.

Then θ^* induces an automorphism $\widehat{\theta}$ of $\widehat{\mathsf{G}^*}$ preserving the fixed Gal_F-invariant pinning of $\widehat{\mathsf{G}}$ [KS99, §1.2]. Set ${}^L\theta = \widehat{\theta} \times \mathrm{id}_{W_F}$, which is an automorphism of ${}^L\mathsf{G}$. Then an *extended endoscopic triple* for (G^*, θ^*) is a triple $\mathfrak{e} = (\mathsf{G}^{\mathfrak{e}}, \mathsf{s}^{\mathfrak{e}}, {}^L\xi^{\mathfrak{e}})$, where $\mathsf{G}^{\mathfrak{e}}$ is a quasisplit reductive group over F, $\mathsf{s}^{\mathfrak{e}} \in \widehat{\mathsf{G}^*}$, and ${}^L\xi^{\mathfrak{e}} : {}^L\mathsf{G}^{\mathfrak{e}} \to {}^L\mathsf{G}^*$ is an L-homomorphism such that

(1) $\mathrm{Ad}(\mathsf{s}^{\mathfrak{e}}) \circ \widehat{\theta}$ preserves a pair of a Borel subgroup and a maximal torus in $\widehat{\mathsf{G}}$, and $\mathrm{Ad}(\mathsf{s}^{\mathfrak{e}}) \circ \widehat{\theta} \circ {}^L\xi^{\mathfrak{e}} = {}^L\xi^{\mathfrak{e}}$,

(2) ${}^L\xi^{\mathfrak{e}}(\widehat{\mathsf{G}^{\mathfrak{e}}})$ is the connected component of the subgroup of $\mathrm{Ad}(\mathsf{s}^{\mathfrak{e}}) \circ \widehat{\theta}$-fixed elements in $\widehat{\mathsf{G}^*}$.

$\mathfrak{e}$ is called *elliptic* if ${}^L\xi^{\mathfrak{e}}(Z(\widehat{\mathsf{G}^{\mathfrak{e}}})^{\mathrm{Gal}_F})^{\circ} \subset Z(\widehat{G^*})$.

Following [KMSW14], we define an isomorphism between two extended endoscopic triples $\mathfrak{e}, \mathfrak{e}'$ to be an element $\mathsf{g} \in \widehat{\mathsf{G}^*}$ such that

$$\mathsf{g}^L\xi^{\mathfrak{e}}({}^L\mathsf{G}^{\mathfrak{e}})\mathsf{g}^{-1} = {}^L\xi^{\mathfrak{e}'}({}^L\mathsf{G}^{\mathfrak{e}'}), \quad g\mathsf{s}^{\mathfrak{e}}\widehat{\theta}(g)^{-1} = \mathsf{s}^{\mathfrak{e}'} (\mathrm{mod}\ \ Z(\widehat{\mathsf{G}})).$$

Denote by $\mathcal{E}(\mathsf{G}^* \rtimes \theta^*)$ the set of isomorphism classes of extended endoscopic triples for (G^*, θ^*). When $\theta^* = \mathrm{id}_{\mathsf{G}^*}$, we also write $\mathcal{E}(\mathsf{G}^*)$ for $\mathcal{E}(\mathsf{G}^* \rtimes \theta^*)$. Also we write $\mathcal{E}_{\mathrm{ell}}(\mathsf{G}^* \rtimes \theta^*)$ or $\mathcal{E}_{\mathrm{ell}}(\mathsf{G}^*)$ for the subset of elliptic extended endoscopic triples.

Suppose $\mathfrak{e} \in \mathcal{E}(\mathsf{G}^* \rtimes \theta^*)$. For each $g^{\mathfrak{e}} \in \widehat{\mathsf{G}}^{\mathfrak{e}}$, the ${}^L\xi^{\mathfrak{e}}(g^{\mathfrak{e}}) \in {}^LG$ induces an automorphism of $\mathfrak{e}$. Define the outer automorphism group of $\mathfrak{e}$ by

$$\mathrm{OAut}(\mathfrak{e}) := \mathrm{Aut}(\mathfrak{e})/{}^L\xi^{\mathfrak{e}}(\widehat{\mathsf{G}^{\mathfrak{e}}}). \tag{A.1}$$

A.2. **Transfer of orbital integrals.** Here we recall some notions on the theory of transfer, following Arthur [Art13, §2.1] and Mok [Mok15, §3.1].

Let K be a local field of characteristic zero, (G^*, θ^*) be a pair as in Appendix §A.1, and (G, ϱ, z) be a pure inner twist of G^*. We get an automorphism $\theta := \varrho \circ \theta^* \circ \varrho^{-1}$, which we assume to be a rational automorphism of G. Moreover, we fix a Whittaker datum $\mathfrak{w}$ for G^*. We write $\mathsf{G} \times \theta$ for the twisted group (or bitorsor) over G. If θ is the identity, then of course $\mathsf{G} \times \theta = \mathsf{G}$ is just the trivial bitorsor. Given $\delta \in \mathsf{G} \times \theta$, we write $Z_{\mathsf{G}}(\delta)$ for the centralizer of δ in G. We write $(\mathsf{G} \times \theta)(K)_{\mathrm{s.reg}} \subset (\mathsf{G} \times \theta)(K)$ for the open subset of strongly regular semisimple elements, meaning those regular semisimple elements whose centralizer is connected, i.e., a maximal torus. We fix a Haar measure on $\mathsf{G}(K)$. For $\delta \in (\mathsf{G} \times \theta)(K)_{\mathrm{s.reg}}$, the Weyl discriminant of δ is defined as

$$D^{\mathsf{G}}(\delta) := \det(1 - \mathrm{Ad}(\delta)|\mathfrak{g}/\mathfrak{g}_\delta) \in K^\times,$$

where $\mathfrak{g}$ and $\mathfrak{g}_\delta$ are the Lie algebras of G and $Z_{\mathsf{G}}(\delta)$, respectively. We fix a Haar measure on the torus $Z_{\mathsf{G}}(\delta)$, which induces a quotient measure on $Z_{\mathsf{G}}(\delta)(K)\backslash\mathsf{G}(K)$.

If K is non-Archimedean, we let $\mathcal{H}(\mathsf{G} \times \theta)$ be the space of smooth compactly supported functions on $(\mathsf{G} \times \theta)(K)$ with complex coefficients. If $K = \mathbb{R}$, we fix a maximal compact subgroup $\mathscr{K}$ of $\mathsf{G}(\mathbb{R})$ and let $\mathcal{H}(\mathsf{G} \times \theta)$ be the space of bi-$\mathscr{K}$-finite smooth compactly supported functions on $(\mathsf{G} \times \theta)(\mathbb{R})$ with complex coefficients.

For $f \in \mathcal{H}(\mathsf{G} \times \theta)$ and $\delta \in (\mathsf{G} \times \theta)(K)_{\mathrm{s.reg}}$, the normalized orbital integral of f along the conjugacy class of δ is defined as

$$\mathrm{Orb}_\delta(f) := \left|D^{\mathsf{G}}(\delta)\right|^{\frac{1}{2}} \int_{Z_{\mathsf{G}}(\delta)(K)\backslash\mathsf{G}(K)} f(x^{-1}\delta x)\mathrm{d}x,$$

Moreover, when G is quasisplit and $\theta^* = \mathrm{id}$, the normalized stable orbital integral of f along δ is defined as

$$\mathrm{SOrb}_\delta(f) := \sum_{\delta'} \mathrm{Orb}_{\delta'}(f),$$

where δ' runs over a set of representatives for the $\mathsf{G}(K)$-conjugacy classes of elements that are $\mathsf{G}(\overline{K})$-conjugate to δ.

Assume that either $(\mathsf{G}, \varrho, z) = (\mathsf{G}^*, \mathrm{id}, \mathbf{1})$ or $\theta = \mathrm{id}$. For $\mathfrak{e} \in \mathcal{E}(\mathsf{G}^* \rtimes \theta^*)$, a *transfer factor*

$$\Delta[\mathfrak{w}, \mathfrak{e}, z] : \mathsf{G}^{\mathfrak{e}}(K)_{\mathrm{s.reg}} \times (\mathsf{G} \times \theta)(K)_{\mathrm{s.reg}} \to \mathbb{C}$$

is defined in [KMSW14, §1.1.2], such that $\Delta[\mathfrak{w}, \mathfrak{e}, z]$ is a function on stable conjugacy classes of $\mathsf{G}^{\mathfrak{e}}(K)_{\mathrm{s.reg}}$ and $\mathsf{G}(K)$-conjugacy classes of $(\mathsf{G} \times \theta)(K)_{\mathrm{s.reg}}$. With the transfer factor in hand, we now recall the notion of matching test functions from [KS99, §5.5].

Definition A.2.1. Two functions $f^{\mathsf{G}^{\mathfrak{e}}} \in \mathcal{H}(\mathsf{G}^{\mathfrak{e}})$ and $f \in \mathcal{H}(\mathsf{G} \times \theta)$ are called ($\Delta[\mathfrak{w}, \mathfrak{e}, z]$-) *matching test functions* if, for every $\gamma \in \mathsf{G}^{\mathfrak{e}}(K)_{\mathrm{s.reg}}$,

$$\mathrm{SOrb}_\gamma(f^{\mathsf{G}^{\mathfrak{e}}}) = \sum_{\delta \in (\mathsf{G} \times \theta)(K)_{\mathrm{s.reg}}/\mathsf{G}(K)\text{-conj}} \Delta[\mathfrak{w}, \mathfrak{e}, z](\gamma, \delta)\, \mathrm{Orb}_\delta(f).$$

For brevity, say that $f^{\mathsf{G}^{\mathfrak{e}}}$ is a transfer of f to $\mathsf{G}^{\mathfrak{e}}$.

Remark A.2.2. Since the orbital integrals $\mathrm{Orb}_\delta(f)$ depend on the choices of measures on $\mathsf{G}(K)$ and $Z_{\mathsf{G}}(\delta)(K)$, the concept of matching functions also depends on the choice of Haar measures on $\mathsf{G}(K)$ and $\mathsf{G}^{\mathfrak{e}}(K)$, and all tori in G and $\mathsf{G}^{\mathfrak{e}}$. There is a way to synchronize the various tori; cf. [AK24, Remark 5.1.2].

We now state a theorem asserting existence of transfer of orbital integrals. When $K = \mathbb{R}$, it is a fundamental result of Shelstad [She82, She08]. When K is non-Archimedean, it is a culmination of the work of many people, including Langlands and Shelstad [LS87, LS90], Waldspurger [Wal97, Wal06], and Ngô [Ngô10].

Theorem A.2.3. *Let $f \in \mathcal{H}(\mathsf{G} \times \theta)$ and $\mathfrak{e} \in \mathcal{E}_{\mathrm{ell}}(\mathsf{G})$. Then there exists a transfer $f^{\mathsf{G}^{\mathfrak{e}}}$ of f to $\mathsf{G}^{\mathfrak{e}}$.*

Moreover, suppose $\mathcal{K} \subset \mathsf{G}(K)$ is a θ-stable hyperspecial maximal compact open subgroup. Then there exists a hyperspecial maximal compact open subgroup $\mathcal{K}^{\mathfrak{e}} \subset \mathsf{G}^{\mathfrak{e}}(K)$ such that the characteristic function $\mathbf{1}_{\mathcal{K}^{\mathfrak{e}}}$ is a transfer of $\mathbf{1}_{\mathcal{K} \times \theta}$ to $\mathsf{G}^{\mathfrak{e}}$, provided the Haar measure is chosen such that

$$\mathrm{Vol}(\mathcal{K}) = \mathrm{Vol}(\mathcal{K}^{\mathfrak{e}}) = 1.$$

A.3. **Cuspidal functions.** In this subsection, we recall the definition of cuspidal and stabilizing functions, following Labesse [Lab99, Definition 3.8.1, 3.8.2]. Recall that an element $\gamma \in \mathsf{G}(K)$ is called an elliptic element if the maximal split subtorus of the center of $Z_{\mathsf{G}}(\gamma)$ is equal to the maximal split subtorus of $Z(\mathsf{G})$.

Definition A.3.1. Suppose K is non-Archimedean of characteristic zero. A function $f \in \mathcal{H}(\mathsf{G})$ is called

- *cuspidal* if the orbital integral of f vanishes on all regular semisimple non-elliptic elements.
- *strongly cuspidal* if the orbital integral of f vanishes outside regular semisimple elliptic elements.
- *stabilizing* if it is cuspidal and the κ-orbital integral of f (as defined in [Lab99, p. 68]) vanishes on all semisimple elements γ and all nontrivial κ.

We recall the notion of pseudo-coefficients of [Kaz86, Clo86]:

Definition A.3.2. Fix a Haar measure on $\mathsf{G}(K)$. For a square-integrable irreducible admissible representation π of $\mathsf{G}(K)$, a function $f_\pi \in \mathcal{H}(\mathsf{G})$ is called a *pseudo-coefficient* for π if

$$\mathrm{tr}(f_\pi | \pi') = \delta_{\pi, \pi'}$$

for each tempered representation π' of $\mathsf{G}(K)$.

Proposition A.3.3 ([Kaz86, Clo86])**.** *Suppose K is non-Archimedean of characteristic zero, and fix a Haar measure on $\mathsf{G}(K)$. For any square-integrable irreducible admissible representation π of $\mathsf{G}(K)$, there exists a pseudo-coefficient f_π for π. Moreover, for any such f_π:*

(1) $\mathrm{tr}(f_\pi|\pi') = 0$ for any finite-length admissible smooth representation π' of $\mathsf{G}(K)$ that is parabolically induced from a proper parabolic subgroup of G.

(2) For every regular elliptic element $\gamma \in \mathsf{G}(K)$,

$$\int_{Z_{\mathsf{G}}(\gamma)(K)\backslash \mathsf{G}(K)} f_\pi(g^{-1}\gamma g)\mathrm{d}g = \Theta_\pi(\gamma),$$

where Θ_π denotes the Harish-Chandra character of π.

(3) f_π is cuspidal.

(4) If π is supercuspidal, then $\mathrm{tr}(f_\pi|\pi') = \delta_{\pi,\pi'}$ for every irreducible admissible representation π' of $\mathsf{G}(K)$.

Proof. The existence of f_π is established by Kazhdan [Kaz86, Theorem K] and Clozel [Clo86, Proposition 1]. The trace values $\Theta_{\pi'}(f_\pi)$ for admissible representations π' of $\mathsf{G}(K)$ and the orbital integrals

$$\int_{Z_{\mathsf{G}}(\gamma)(K)\backslash \mathsf{G}(K)} f_\pi(g^{-1}\gamma g)\mathrm{d}g$$

for any regular elliptic elements $\gamma \in \mathsf{G}(K)$ are independent of the choice of f_π by [Kaz86, Theorem 0]. Assertions (i) and (iii) are built into Kazhdan's construction [Kaz86, Theorem K, Theorem A.(b)], and (ii) follows from [Kaz86, Theorem K].

For (iv), if π' is supercuspidal, the assertion follows from the defining property of a pseudo-coefficient, since supercuspidal representations are tempered. If π' is not supercuspidal, then its image in the Grothendieck group of finite-length admissible representations is a linear combination of representations induced from proper parabolic subgroups of G [Clo86, Proposition 2]; hence (i) gives $\mathrm{tr}(f_\pi|\pi') = 0$. □

A.4. **Simple stable trace formulas.** Here we recall some results on simple stable trace formulas from [KS23, Ham25]. We work in the following setting.

Notation A.4.1.

- F is a totally real number field.
- $\mathbb{G}^*$ is a quasisplit reductive group over F that is simple over $\overline{F}$; assume $\mathbb{G}(F \otimes \mathbb{R})$ admits discrete series.
- $(\mathbb{G}, \varrho, z)$ is a pure inner twist of $\mathbb{G}^*$.
- Let Z denote the center of $\mathbb{G}^*$, and let A_Z denote the maximal split torus of $\mathrm{Res}_{F/\mathbb{Q}}Z$; write $A_{Z,\infty} := A_Z(\mathbb{R})^\circ$.
- Let $\mathscr{K}_\infty = \prod_{v\in\Sigma_F^\infty}\mathscr{K}_v \le \mathbb{G}(F\otimes\mathbb{R})$ be the product of a maximal compact subgroup with $Z_{\mathbb{G}}(F\otimes\mathbb{R})$.
- For each finite place v of F, set $q(\mathbb{G}_v)$ to be the F_v-rank of $\mathbb{G}_{v,\mathrm{ad}}$.
- For each infinite place v of F, set $q(\mathbb{G}_v)$ to be the real dimension of the locally symmetric space $\mathbb{G}(F_v)/\mathscr{K}_v$.
- Set

  $$\mathbb{G}(\mathbf{A}_F)^1 := \bigcap_{\chi\in X^*(\mathbb{G})} \ker(\|-\|\circ\chi : \mathbb{G}(\mathbf{A}_F) \to \mathbb{R}_+).$$

  In particular, $\mathbb{G}(\mathbf{A}_F) = \mathbb{G}(\mathbf{A}_F)^1 \times A_{Z,\infty}$.

Definition A.4.2. A *central character datum* for $\mathbb{G}$ is a pair $(\mathfrak{X},\chi)$ where $\mathfrak{X}$ is a closed subgroup of $Z(\mathbf{A}_F)$ containing $A_{Z,\infty}$ such that $Z(F)\mathfrak{X}$ is closed in $Z(\mathbf{A}_F)$, and $\chi : (\mathfrak{X}\cap Z(F))\backslash\mathfrak{X} \to \mathbb{C}^\times$ is a continuous character. In particular, $Z(F)\mathfrak{X}$ is cocompact in $Z(\mathbf{A}_F)$ because $Z(F)\backslash Z(\mathbf{A}_F)/A_{Z,\infty}$ is compact.

For our purposes, it suffices to consider the cases when $\mathfrak{X} = \prod_{v\in\Sigma_F}\mathfrak{X}_v$ where $\mathfrak{X}_\tau = Z(F_\tau)$ for each $\tau\in\Sigma_F^\infty$.

Note that the center of $\mathbb{G}$ is isomorphic to Z via ϱ, so any central character datum for $\mathbb{G}^*$ may be regarded as a central character datum for $\mathbb{G}$.

Remark A.4.3. We suppress the choice of Haar measures for various groups below as they are standard.

Definition A.4.4. Given a central character datum $(\mathfrak{X},\chi)$ for $\mathbb{G}^*$ of the form $\mathfrak{X} = \prod_{v\in\Sigma_F}\mathfrak{X}_v$, for each $v\in\Sigma_F$, let $\mathcal{H}(\mathbb{G}(F_v),\chi_v^{-1})$ be the space of smooth functions on $\mathbb{G}(F_v)$ that are compactly supported modulo center and transform under $\mathfrak{X}_v$ by the character χ_v^{-1} and moreover $\mathscr{K}_v$-finite when $v\in\Sigma_F^\infty$.

Given a semisimple element $\gamma_v\in\mathbb{G}(F_v)$ with $I_{\gamma_v} = Z_{\mathbb{G}_{F_v}}(\gamma_v)^\circ$, we define the orbital integral on $\mathcal{H}(\mathbb{G}(F_v),\chi_v^{-1})$ to be

$$\mathrm{Orb}_{\gamma_v}(f_v) := \int_{I_{\gamma_v}(F_v)\backslash\mathbb{G}(F_v)} f_v(x_v^{-1}\gamma_v x_v)\mathrm{d}x_v,$$

where $I_{\gamma_v}\backslash\mathbb{G}(F_v)$ is given the Euler–Poincaré measure as defined in [Kot88, §1]. If π_v is an admissible representation of $\mathbb{G}(F_v)$ with central character χ_v on $\mathfrak{X}_v$, we define the trace character on $\mathcal{H}(\mathbb{G}(F_v),\chi_v^{-1})$ to be

$$\Theta_{\pi_v}(f_v) := \mathrm{tr}\Big(\int_{\mathbb{G}(F_v)/\mathfrak{X}_v} f_v(g)\pi_v(g)\mathrm{d}g\Big).$$

We also define the adelic Hecke algebra $\mathcal{H}(\mathbb{G}(\mathbf{A}_F),\chi^{-1})$ as well as adelic orbital integrals and adelic trace characters by taking restricted tensor product over the local cases considered above.

Definition A.4.5. Given a central character datum $(\mathfrak{X},\chi)$ for $\mathbb{G}^*$, we write

- $\Gamma_{\mathrm{ell},\mathfrak{X}}(\mathbb{G})$ for the set of $\mathfrak{X}$-orbits of elliptic conjugacy classes in $\mathbb{G}(F)$,
- $\Sigma_{\mathrm{ell},\mathfrak{X}}(\mathbb{G})$ for the set of $\mathfrak{X}$-orbits of elliptic stable conjugacy classes in $\mathbb{G}(F)$,
- $L^2_{\mathrm{disc},\chi}(\mathbb{G}(F)\backslash\mathbb{G}(\mathbf{A}_F))$ for the space of measurable functions on $\mathbb{G}(F)\backslash\mathbb{G}(\mathbf{A}_F)$ transforming under $\mathfrak{X}$ by χ and square-integrable on $\mathbb{G}(F)\backslash\mathbb{G}(\mathbf{A}_F)^1/\left(\mathfrak{X}\cap\mathbb{G}(\mathbf{A}_F)^1\right)$,
- $L^2_{\mathrm{cusp},\chi}(\mathbb{G}(F)\backslash\mathbb{G}(\mathbf{A}_F))$ for the space of cuspidal measurable functions on $\mathbb{G}(F)\backslash\mathbb{G}(\mathbf{A}_F)$ transforming under $\mathfrak{X}$ by χ and square-integrable on $\mathbb{G}(F)\backslash\mathbb{G}(\mathbf{A}_F)^1/\left(\mathfrak{X}\cap\mathbb{G}(\mathbf{A}_F)^1\right)$,
- $\mathrm{Irr}^{\mathrm{cusp}}_\chi(\mathbb{G})$ the set of isomorphism classes of cuspidal automorphic representations of $\mathbb{G}(\mathbf{A}_F)$ whose central characters under $\mathfrak{X}$ are χ.

We define the following invariant distributions on $\mathcal{H}(\mathbb{G}(\mathbf{A}_F),\chi^{-1})$:

$$\mathrm{T}^{\mathbb{G}}_{\mathrm{ell},\chi}(f) := \sum_{\gamma\in\Gamma_{\mathrm{ell},\mathfrak{X}}(\mathbb{G})}\frac{1}{\#\pi_0(Z_{\mathbb{G}}(\gamma))}\mathrm{Vol}\left(I_\gamma(F)\backslash I_\gamma(\mathbf{A}_F)/\mathfrak{X}\right)\mathrm{Orb}_\gamma(f),$$

$$\mathrm{T}^{\mathbb{G}}_{\mathrm{disc},\chi}(f) := \mathrm{tr}\left(f|L^2_{\mathrm{disc},\chi}(\mathbb{G}(F)\backslash\mathbb{G}(\mathbf{A}_F))\right),$$

$$\mathrm{T}^{\mathbb{G}}_{\mathrm{cusp},\chi}(f) := \mathrm{tr}\left(f|L^2_{\mathrm{cusp},\chi}(\mathbb{G}(F)\backslash\mathbb{G}(\mathbf{A}_F))\right),$$

Next, we recall the definitions of unramified twists of Steinberg representations and Lefschetz functions.

Definition A.4.6. The Steinberg representation $\mathrm{St}_{\mathbb{G}_v}$ is the discrete series representation defined in [BW00, 10.4.6]. An unramified twist of $\mathrm{St}_{\mathbb{G}_v}$ is the twist of the Steinberg representation by an unramified character of $\mathbb{G}(F_v)$, where a character of $\mathbb{G}(F_v)$ is unramified if it is trivial on all compact subgroups of $\mathbb{G}(F_v)$; see [Cas95, p. 17].

By [Kot88, Theorem 2 and Theorem 2'] and [KS23, Proposition A.1, A.4 and Lemma A.7], we introduce the following non-Archimedean Lefschetz function:

Definition A.4.7. Let v be a finite place of F. If Z_{F_v} is anisotropic, there exists a *Lefschetz function* $f^{\mathbb{G}}_{\mathrm{Lef},v} \in \mathcal{H}(\mathbb{G}(F_v))$ such that

- If $\mathbb{G}_{\mathrm{ad}}$ is simple, then for each irreducible admissible representation π of $\mathbb{G}(F_v)$,
$$\mathrm{tr}\left(f^{\mathbb{G}}_{\mathrm{Lef},v}|\pi\right) = \begin{cases} 1 & \pi = \mathbf{1}, \\ (-1)^{q(\mathbb{G}_v)} & \pi = \mathrm{St}_{\mathbb{G}_v}, \\ 0 & \text{otherwise.} \end{cases}$$
- If $\gamma_v \in \mathbb{G}(F_v)$ is semisimple with $I_{\gamma_v} = Z_{\mathbb{G}_v}(\gamma_v)^\circ$, then the orbital integral
$$\mathrm{Orb}_{\gamma_v}(f^{\mathbb{G}}_{\mathrm{Lef},v}) = \int_{I_{\gamma_v}(F_v)\backslash\mathbb{G}(F_v)} f^{\mathbb{G}}_{\mathrm{Lef},v}(g_v^{-1}\gamma_v g_v)\mathrm{d}g_v$$
vanishes unless $Z(I_{\gamma_v}(F_v))$ is compact, in which case $\mathrm{Orb}_{\gamma_v}(f^{\mathbb{G}}_{\mathrm{Lef},v}) = 1$. Here $I_{\gamma_v}(F_v)\backslash\mathbb{G}(F_v)$ is endowed with the Euler–Poincaré measure.

In general, set A_v to be the maximal split torus in the center of $\mathbb{G}_v$, and set $\mathbb{G}'_v := \mathbb{G}_v/A_v$. Let
$$\nu : \mathbb{G}(F_v) \to X^\bullet(A_v) \otimes \mathbb{R}$$
denote the valuation map as in [Lab99, §3.9], with kernel $\mathbb{G}(F_v)^1$. We define
$$f^{\mathbb{G}_v}_{\mathrm{Lef},v} := \mathbf{1}_{\mathbb{G}(F_v)^1} \cdot f^{\mathbb{G}'_v}_{\mathrm{Lef},v}.$$
Then:

- $f^{\mathbb{G}}_{\mathrm{Lef},v}$ is strongly cuspidal and stabilizing (see Definition A.3.1);
- if $\mathrm{tr}(f^{\mathbb{G}}_{\mathrm{Lef},v}|\pi_v) \neq 0$ for some irreducible unitary representation π_v of $\mathbb{G}(F_v)$, then π_v is an unramified character twist of either the trivial representation or the Steinberg representation;
- $(-1)^{q(\mathbb{G}_v)} f^{\mathbb{G}}_{\mathrm{Lef},v}$ and $(-1)^{q(\mathbb{G}^*_v)} f^{\mathbb{G}^*}_{\mathrm{Lef},v}$ are associated.

Assume that $\mathbb{G}(F\otimes\mathbb{R})$ admits discrete series. We introduce the following Archimedean Lefschetz function.

Definition A.4.8. Suppose $\tau \in \Sigma_F^\infty$ and ξ_τ is an irreducible algebraic representation of $\mathbb{G}_\tau$ with regular highest weight. Denote by $\chi_{\xi_\tau} : Z(F_\tau) \to \mathbb{C}^\times$ the inverse of the central character of ξ_τ. Then there exists a *Lefschetz function*
$$f^{\mathbb{G}}_{\xi_\tau} \in \mathcal{H}(\mathbb{G}(F_\tau), \chi_{\xi_\tau}^{-1})$$
associated with ξ_τ such that
$$\mathrm{tr}(f_{\xi_\tau}|\pi_\tau) = \mathrm{ep}_{\mathcal{K}_\tau}(\pi_\tau \otimes \xi_\tau) := \sum_{i\in\mathbb{N}}(-1)^i \dim \mathrm{H}^i(\mathrm{Lie}(\mathbb{G}_\tau(\mathbb{R})), \mathcal{K}_\tau; \pi_\tau \otimes \xi_\tau)$$
for each irreducible admissible representation π_τ of $\mathbb{G}_\tau(\mathbb{R})$ whose central character equals the inverse of the central character of ξ_τ.

For any irreducible admissible $(\mathfrak{g}_\tau, \mathcal{K}_\tau)$-module π_τ such that $\mathrm{tr}\big(f^{\mathbb{G}}_{\xi_\tau}|\pi_\tau\big) \neq 0$, the representation π_τ is a discrete series representation cohomological for ξ_τ; equivalently, π_τ has the same central character and infinitesimal character as $\xi^\vee$. Moreover,
$$\mathrm{tr}\big(f^{\mathbb{G}}_{\xi_\tau}|\pi_\tau\big) = (-1)^{q(\mathbb{G}_\tau)}.$$
This follows from the Vogan–Zuckerman classification of unitary cohomological representations; cf. [Shi12, Lemma 2.7].

Definition A.4.9. Given a central character datum $(\mathfrak{X}, \chi)$ for $\mathbb{G}^*$, we define the stably invariant distributions on $\mathcal{H}(\mathbb{G}^*(\mathbf{A}_F), \chi^{-1})$ by

$$\mathrm{ST}^{\mathbb{G}^*}_{\mathrm{ell},\chi}(f^*) := \tau(\mathbb{G}^*) \sum_{\gamma \in \Sigma_{\mathrm{ell},\mathfrak{X}}(\mathbb{G}^*)} \frac{1}{\#\left(\pi_0(Z_{\mathbb{G}^*}(\gamma))\right)^{\mathrm{Gal}_F}} \mathrm{SOrb}^{\mathbb{G}^*}_{\gamma,\chi}(f^*),$$

where $\tau(\mathbb{G}^*)$ is the Tamagawa number of $\mathbb{G}^*$ and

$$\mathrm{SOrb}^{\mathbb{G}^*}_{\gamma,\chi}(f^*) = \sum_{\gamma'} \mathrm{Orb}^{\mathbb{G}^*}_{\gamma',\chi}(f^*)$$

is the stable orbital integral at γ, with γ' running through a set of representatives for the F-conjugacy classes inside the stable conjugacy class of γ.

Theorem A.4.10. *Fix a central character datum $(Z(F \otimes \mathbb{R}), \chi)$, where χ is the inverse of the central character of an irreducible representation*

$$\xi = \boxtimes_{\tau \in \Sigma_F^\infty} \xi_\tau$$

of $\mathbb{G}_\mathbb{C}$ with regular highest weight, restricted to $Z(F \otimes \mathbb{R})$. Suppose $f \in \mathcal{H}(\mathbb{G}(\mathbf{A}_F), \chi^{-1})$ satisfies

- *$f_\infty = \bigotimes_{\tau \in \Sigma_F^\infty} f^{\mathbb{G}}_{\xi_\tau}$ is an Archimedean Lefschetz function attached to ξ;*
- *there exists a finite place v of F such that $f_v = f^{\mathbb{G}}_{\mathrm{Lef},v}$ is a non-Archimedean Lefschetz function.*

Then

$$\mathrm{ST}^{\mathbb{G}^*}_{\mathrm{ell},\chi}(f^*) = \mathrm{T}^{\mathbb{G}}_{\mathrm{ell},\chi}(f) = \mathrm{T}^{\mathbb{G}}_{\mathrm{disc},\chi}(f) = \mathrm{T}^{\mathbb{G}}_{\mathrm{cusp},\chi}(f),$$

where $f^* \in \mathcal{H}(\mathbb{G}^*(\mathbf{A}_F), \chi^{-1})$ is a transfer of f to $\mathbb{G}^*$.

Proof. This follows from [KS23, Lemma 6.1, 6.2]. □

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