Notes on arithmetic intersection theory

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Contents

Let F/F_0 be a CM extension of a totally real number field, and let X_K be a unitary Shimura variety canonically defined over F arising from a totally definite incoherent hermitian space $\mathbb{V}$ over $\mathbb{A}_F$.

In Kudla's paper, he proposed that the modularity problem of arithmetic theta functions should take place in the *arithmetic Chow group* $\widehat{\mathrm{CH}}^n(\mathcal{X}_K)$ of some suitable (compactified) regular integral model $\mathcal{X}_K$ of (a variant of) X_K, where *arithmetic intersection theory* takes place. The latter is what this talk will address.

This material is heavily drawn from [GS90], and [Sou21].

1 Notions from complex geometry

1.1 Differential forms

Let X be a smooth complex manifold of dimension d, and $\mathcal{O}_X$ its sheaf of holomorphic functions. Denote by $A^n(X)$ the space of C^∞ complex-valued n-forms on X (as a real manifold). Recall that $A^n(X)$ decomposes as

$$A^n(X) = \bigoplus_{p+q=n} A^{p,q}(X),$$

where $A^{p,q}$ consists of those differential forms which can be written locally as a sum of forms of type

$$u dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\overline{z}_{j_1} \wedge \cdots \wedge d\overline{z}_{j_q}$$

where u is a C^∞ function, $dz_\alpha = dx_\alpha + i dy_\alpha$, and $d\overline{z}_\alpha = dx_\alpha - i dy_\alpha$.

The exterior derivative

$$d : A^n(X) \to A^{n+1}(X)$$

is a sum $d = \partial + \overline{\partial}$ where

$$\partial : A^{p,q}(X) \to A^{p+1,q}(X)$$
$$\overline{\partial} : A^{p,q}(X) \to A^{p,q+1}(X).$$

We have $\partial^2 = \overline{\partial}^2 = d^2 = 0$ and we let

$$d^c = \frac{\partial - \overline{\partial}}{4\pi i},$$

so that

$$dd^c = \frac{\overline{\partial}\partial}{2\pi i}.$$

1.2 Line bundles with metric

Definition 1.1. Let L be a holomorphic line bundle on X (a locally free $\mathcal{O}_X$-module of rank one). A *Hermitian metric* $\|\cdot\|$ on L consists of maps

$$L(x) \to \mathbb{R}_{>0}$$

for any $x \in X$, where $L(x) = L_x/\mathfrak{m}_x L_x$ is the fiber at x. We require that

(1) $\|\lambda s\| = |\lambda|\|s\|$ for all $\lambda \in \mathbb{C}$,

(2) $\|s\| = 0$ iff $s = 0$;

(3) If $U \subset X$ an open subset and s a section of L over U vanishing nowhere; then the map

$$x \mapsto \|s(x)\|^2$$

is C^∞.

(4) the metric $\|\cdot\|$ is Hermitian, i.e. invariant under complex conjugation $F_\infty : X(\mathbb{C}) \to X(\mathbb{C})$.

We write $\overline{L} = (L, \|\cdot\|)$ for a Hermitian line bundle.

Lemma 1.2. *Let $\overline{L} = (L, \|\cdot\|)$ be a Hermitian line bundle. There exists a smooth form*

$$c_1(\overline{L}) \in A^{1,1}(X)$$

such that, if $U \subset X$ is an open subset and $s \in \Gamma(U, L)$ is such that $s(x) \neq 0$ for every $x \in U$,

$$c_1(\overline{L})|_U = -dd^c \log \|s\|^2.$$

Proof. If $s' \in \Gamma(U, L)$ is another nowhere-vanishing section, then there exists $f \in \Gamma(U, \mathcal{O}_X)$ such that

$$s' = fs.$$

We get

$$-dd^c \log \|s\|^2 = -dd^c \log \|s\|^2 - dd^c \log |f|^2.$$

But

$$\partial\overline{\partial} \log |f|^2 = \partial \left[\frac{\overline{\partial} f}{f} + \frac{\overline{\partial f}}{\overline{f}} \right] = -\overline{\partial}\frac{\partial(\overline{f})}{\overline{f}} = 0,$$

and thus $c_1(\overline{L})|_U$ is well-defined. $\square$

We then call $c_1(\overline{L})$ the *first Chern form* of $\overline{L}$.

1.3 Integration

We need to make sense of integration of forms along closed subsets, which will draw upon Hironaka's resolution theorem:

> **Theorem 1.3** (Hironaka). *Let Y be a scheme of finite type over $\mathbb{C}$, and $Z \subset Y$ a proper closed subset of Y such that $Y \setminus Z$ is smooth. Then there exists a proper map*
>
> $$\pi : \widetilde{Y} \to Y$$
>
> *such that*
>
> (1) $\widetilde{Y}$ *is smooth;*
>
> (2) $\widetilde{Y} \setminus \pi^{-1}(Z) \xrightarrow{\sim} Y \setminus Z$;
>
> (3) $\pi^{-1}(Y)$ *is a divisor with normal crossings.*

Suppose Y was some integral closed subset of a smooth projective variety over $\mathbb{C}$, and s is some nontrivial rational section of that variety over Y. Applying Hironaka's theorem to Y and the union $Z = \mathrm{supp}(\mathrm{div}(s)) \cup Y(\mathbb{C})^{\mathrm{sing}}$ of the support of $\mathrm{div}(s)$ and the singular locus of $Y(\mathbb{C})$. Let $\pi : \widetilde{Y} \to Y$ the resolution of Y, d the dimension of $Y(\mathbb{C})$, and $\omega \in A_c^{d,d}(Y(\mathbb{C}) \setminus Z)$ of compact support. Then we define

$$\int_{Y(\mathbb{C})} \log\|s\|\omega = \int_{\widetilde{Y}(\mathbb{C})} \log\|\pi^*(s)\|\pi^*(\omega).$$

One shows that this integral converges.

1.4 Poincaré–Lelong formula

We often assume our manifolds are compact or something.

> **Proposition 1.4** (Poincaré–Lelong). *Let $\overline{L} = (L, \|\cdot\|)$ be a Hermitian line bundle on X, and s a meromorphic section of L. Then we have the following formula in $D^{1,1}(X)$:*
>
> $$dd^c(-\log \|s\|^2) + \delta_{\mathrm{div}(s)} = c_1(\overline{L}).$$

Proof. Let $Z = \mathrm{supp}(\mathrm{div}(s))$. Let $\pi : \widetilde{X} \to X$ be a resolution where $\pi^{-1}(Z)$ has local equation $z_1 \cdots z_k = 0$. Therefore

$$\pi^*(s) = z_1^{n_1} \cdots z_k^{n_k}$$

locally. By proving the claim for $\pi^*(\overline{L})$ and $\pi^*(s)$ and applying π_*, we may assume that $\widetilde{X} = X$. By additivity we can assume

(a) $\|s\| = |z_1|$ or

(b) $\log\|s\| = \rho \in C^\infty(X)$.

In case (b), $\mathrm{div}(s) = 0$ and the formula is true by definition of $c_1(\overline{L})$. In case (a) we have to show that, for every differential form ω with compact support in U, and for $\epsilon > 0$ small enough,

$$-\int_U \log|z_1|^2 dd^c(\omega) = \int_{|z_1|=\epsilon} \omega.$$

$\square$

1.5 Currents

Definition 1.5. A *current* on the complex manifold X is a $\mathbb{C}$-linear functional

$$T : A^{d-p,d-q}(X) \to \mathbb{C}$$

which is continuous for the Schwartz topology.

Example 1.6.

(1) If $\eta \in L^1(X) \otimes_{C^\infty(X)} A^{p,q}(X)$ is an integrable differential, η defines a current by the formula

$$\eta(\omega) = \int_X (\eta \wedge \omega).$$

(2) If $Z = \sum_{i=1}^m n_i Z_i$ is an (analytic) cycle of codimension p on X, it defines a Dirac current $\delta_Z \in D^{p,p}(X)$ by the formula

$$\delta_Z(\omega) = \sum_{i=1}^m n_i \int_{Z_i} \omega,$$

where the integrals converge by Hironaka's theorem.

We can derivate a current $T \in D^{p,q}(X)$ by the formula

$$\partial T(\omega) = (-1)^{p+q+1} T(\partial\omega)$$

and

$$\overline{\partial} T(\omega) = (-1)^{p+q+1} T(\overline{\partial}\omega).$$

By Stokes' formula, we get commutative diagrams

$$\begin{array}{ccc} D^{p,q}(X) & \xrightarrow{\partial} & D^{p+1,q}(X) \\ \cup & & \cup \\ A^{p,q}(X) & \xrightarrow{\partial} & A^{p+1,q}(X) \end{array} \quad \text{and} \quad \begin{array}{ccc} D^{p,q}(X) & \xrightarrow{\overline{\partial}} & D^{p,q+1}(X) \\ \cup & & \cup \\ A^{p,q}(X) & \xrightarrow{\overline{\partial}} & A^{p,q+1}(X) \end{array}$$

We let

$$\widetilde{A}^{p,q}(X) = A^{p,q}/(\partial A^{p-1,q}(X) + \overline{\partial} A^{p,q-1}(X))$$

and

$$\widetilde{D}^{p,q}(X) = D^{p,q}/(\partial D^{p-1,q}(X) + \overline{\partial} D^{p,q-1}(X)),$$

so that $dd^c : D^{p,q}(X) \to D^{p+1,q+1}(X)$ factors through a map $dd^c : \widetilde{D}^{p,q}(X) \to D^{p+1,q+1}(X)$.

It is known that the d, ∂, and $\overline{\partial}$ cohomology of currents is the same as the d, ∂, and $\overline{\partial}$ cohomology of C^∞ forms, which can be used to obtain a further description.

Proposition 1.7. *We have the following.*

(1) *The natural map* $\widetilde{A}^{p,q}(X) \to \widetilde{D}^{p,q}(X)$ *is an injection.*

(2) *For all* $T \in \widetilde{D}^{p,q}(X)$,
$$dd^c T \in A^{p+1,q+1}(X) \Rightarrow T \in \widetilde{A}^{p,q}(X).$$

(3) *If* X *is compact and Kähler, the restriction of the canonical map* $D^{p,q}(X) \to \widetilde{D}^{p,q}(X)$ *to the space* $H^{p,q}(X)$ *of harmonic forms is injective, its image is the kernel of* dd^c*, and the image of* dd^c *consists of the* d*-exact currents in* $D^{p+1,q+1}(X)$.

Proof. See [GS90, Theorem 1.2.2, 1.2.3]. □

2 Arithmetic intersection theory over $\mathbb{Z}$

Let X be a regular projective flat scheme over $\mathbb{Z}$ and $p \geq 0$ an integer. Let $Z^p(X)$ be the group of cycles of codimension p on X (i.e. the free abelian group generated by integral subschemes of X of codimension p). Recall that the Chow group is defined as

$$\mathrm{CH}^p(X) = Z^p(X)/R^p(X),$$

where $R^p(X)$ is the group of cycles rationally equivalent to 0.

Fact. *If Y is a codimension p integral subscheme of X, then $Y(\mathbb{C})$ is an analytic subspace of $X(\mathbb{C})$. Thus, it makes sense to talk about Green currents for algebraic cycles.*

2.1 Definition of the arithmetic Chow group

The first step is to define the Green current. Note that we could also have made this definition in the complex geometric discussion (i.e. for analytic cycles in general), but I chose to present it here for simplicity.

Definition 2.1. If X is a scheme over $\mathbb{Z}$ as above, and $Y = \sum_{i=1}^m n_i Y_i \in Z^p(X)$ is a cycle of codimension p on X, a *Green current* for Y is an element $[g] \in \widetilde{D}^{p-1,p-1}(X(\mathbb{C}))$, such that

(1) $F_\infty^*(g) = (-1)^{p-1} g$, where $F_\infty : X(\mathbb{C}) \to X(\mathbb{C})$ is the antiholomorphic involution induced by complex conjugation;

(2) $dd^c g + \delta_{Y(\mathbb{C})}$ is a C^∞ form $\omega \in A^{p,p}(X(\mathbb{C}))$. Here δ_Y is the Dirac current $\delta_{Y(\mathbb{C})} \in D^{p,p}(X(\mathbb{C}))$ as in Example 1.6.

One also works with the notion of a *Green form* for Y, which is an element $g \in A^{p-1,p-1}(X(\mathbb{C}))$ (i.e. a Green current up to $\partial D^{p-2,q-1}(X(\mathbb{C})) + \bar{\partial} D^{p-1,q-2}(X(\mathbb{C}))$), locally L^1 on $X(\mathbb{C})$ and C^∞ on $X(\mathbb{C}) \setminus \mathrm{supp}(Y(\mathbb{C}))$, such that its image $[g] \in \widetilde{D}^{p-1,p-1}(X(\mathbb{C}))$ is a Green current for Y.

Theorem 2.2. *Any algebraic cycle on a smooth quasi-projective variety over $\mathbb{C}$ admits a Green form (of* log type*, i.e. a form coming from a resolution of singularities which is logarithmic at the singular locus).*

Definition 2.3. The group $\widehat{Z}^p(X)$ of *arithmetic cycles of codimension p* is the subgroup of

$$Z^p(X) \oplus \widetilde{D}^{p-1,p-1}(X(\mathbb{C}))$$

consisting of pairs $(Z = \sum n_i [Z_i], g)$ such that g is a Green current for $Z(\mathbb{C})$ on the complex manifold $X(\mathbb{C})$, with $(Z_1, g_1) + (Z_2, g_2) = (Z_1 + Z_2, g_1 + g_2)$.

Example 2.4.

(1) Let $Y \subset X$ be an integral subscheme of codimension $p-1$, and $f \in \kappa(Y)$ a rational function on Y. Define $\log |f|^2 \in D^{p-1,p-1}(X(\mathbb{C}))$ by the formula

$$(\log |f|^2)(\omega) = \int_{Y(\mathbb{C})} \log |f|^2 \omega$$

We may think of f as a rational section of the trivial line bundle on Y, so $\mathrm{div}(f)$ is a codimension

p cycle on X. By the Poincaré–Lelong formula,

$$dd^c(-\log|f|^2) + \delta_{\mathrm{div}(f)} = 0.$$

In this way, we have defined an arithmetic cycle

$$\widehat{\mathrm{div}}(f) = (\mathrm{div}(f), -\log|f|^2),$$

an element of $\widehat{Z}^p(X)$.

(2) We can check that, given $u \in D^{p-2,p-1}(X(\mathbb{C}))$ and $v \in D^{p-1,p-2}(X(\mathbb{C}))$ we have

$$dd^c(\partial u + \overline{\partial} v) = 0,$$

so $\partial u + \overline{\partial} v$ is a Green form for the empty cycle which becomes trivial as a Green current.

Green forms which are trivial as Green currents as in example (2) above seem to lead to some nuisancy in that some papers quotient out those currents when defining $\widehat{Z}^p(X)$ and some papers quotient them out when defining $\widehat{R}^p(X)$.

Definition 2.5. Let $\widehat{R}^p(X) \subset \widehat{Z}^p(X)$ be the subgroup generated by elements of the form $\widehat{\mathrm{div}} f$. Then, the *arithmetic Chow group* of codimension p of X is the quotient

$$\widehat{\mathrm{CH}}^p(X) = \widehat{Z}^p(X)/\widehat{R}^p(X).$$

There are several maps involving $\widehat{\mathrm{CH}}^*(X)$.

- $\zeta : \widehat{\mathrm{CH}}^p(X) \to \mathrm{CH}^p(X)$, $[(Z, g)] \mapsto [Z]$.
- $\omega : \widehat{\mathrm{CH}}^p(X) \to \ker d \cap \ker d^c \subset A^{p,p}(X_{\mathbb{R}})$, $[(Z, g)] \mapsto dd^c g + \delta_{Z(\mathbb{C})}$.

Finally, we let

$$\widehat{\mathrm{CH}}^*(X) = \bigoplus_{p\geq 0} \widehat{\mathrm{CH}}^p(X).$$

2.2 Arithmetic Chow group in codimension 1

In codimension 1, the group $\widehat{\mathrm{CH}}^1(X)$ is the same as the group $\widehat{\mathrm{Pic}}(X)$ of isometry classes of Hermitian line bundles on X. (In this setting where $X/\mathbb{Z}$ was a smooth scheme, a line bundle with metric $\overline{L} = (L, \|\cdot\|)$ is *Hermitian* if its metric is invariant under complex conjugation $F_\infty : X(\mathbb{C}) \to X(\mathbb{C})$.)

If $\overline{L} = (L, \|\cdot\|) \in \widehat{\mathrm{Pic}}(X)$ and if $s \neq 0$ is a rational section of L, we let

$$\widehat{\mathrm{div}}(s) = (\mathrm{div}(s), -\log\|s\|^2) \in \widehat{Z}^1(X)$$

and define $\widehat{c}_1(\overline{L}) \in \widehat{\mathrm{CH}}^1(X)$ to be the class of $\widehat{\mathrm{div}}(s)$. It's simple to check that this does not depend on the choice of s.

Proposition 2.6. *The map $\widehat{c}_1$ induces a group isomorphism*

$$\widehat{c}_1 : \widehat{\mathrm{Pic}}(X) \to \widehat{\mathrm{CH}}^1(X).$$

Proof. Consider the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & C^\infty(X(\mathbb{C})) \arrow[r, "a'"] \arrow[d, equal] & \widehat{\mathrm{Pic}}(X) \arrow[r, "\zeta'"] \arrow[d, "\widehat{c}_1"] & \mathrm{Pic}(X) \arrow[r] \arrow[d, "c_1"] & 0 \\
0 \arrow[r] & C^\infty(X(\mathbb{C})) \arrow[r, "a"] & \widehat{\mathrm{CH}}^1(X) \arrow[r, "\zeta"] & \mathrm{CH}^1(X) \arrow[r] & 0
\end{tikzcd}$$

where $a'(\varphi)$ is the trivial line bundle on X equipped with the norm such that $\|1\| = \exp(\varphi)$, $\zeta'(\overline{L}) = L$, $a(\varphi) = (0, \varphi)$, and $\zeta(Z, g) = Z$. □

2.3 Intersection pairing on Chow groups

Before we state the intersection pairing for arithmetic Chow groups, we recall several things about cycles.

If X is a Noetherian scheme, then

$$Z^p(X) = \bigoplus_{Z \subset X \text{ such that } \operatorname{codim}_X(Z) = p} \mathbb{Z}.$$

If $Y \subset X$ is a closed subset, then

$$Z^p_Y(X) = \bigoplus_{Z \subset Y \text{ such that } \operatorname{codim}_X(Z) = p} \mathbb{Z}.$$

Define $\mathrm{CH}^p_Y(X)$ to be the *Chow group of codimension p cycles with supports in Y*, i.e. the quotient of $Z^p_Y(X)$ by the subgroup generated by cycles of the form $\operatorname{div}(f)$, for $f \in k(W)^\times$ with W a codimension $p-1$ integral subscheme of X contained in Y. There is an exact sequence

$$\mathrm{CH}^p_Y(X) \to \mathrm{CH}^p(X) \to \mathrm{CH}^p(X \setminus Y) \to 0.$$

For instance, $\mathrm{CH}^p_X(X) = \mathrm{CH}^p(X)$.

Now if X is a regular Noetherian scheme, there is an isomorphism

$$\mathrm{CH}^p_Y(X)_{\mathbb{Q}} \simeq K_0(X)_{\mathbb{Q}}^{\psi^k = k^p},$$

where $K_0(X)$ is the Grothendieck group of vector bundles on X. This comes from interpreting $\mathrm{CH}^p_Y(X)_{\mathbb{Q}}$ as the subspace of $K_0^Y(X)_{\mathbb{Q}}$ where the Adams operations ψ^k act by multiplication by k^p ($k \geq 1$). It sends the class $[Z]$ to the class $[\mathcal{O}_Z]$ of a resolution in $C_Y(X)$ (complexes of $\mathcal{O}_X$-modules with homology supported on Y) of the coherent module $\mathcal{O}_Z$.

> **Theorem 2.7.** *If X is a regular Noetherian scheme of finite Krull dimension and Y, Z are closed subsets, there is a pairing*
>
> $$\mathrm{CH}^p_Y(X) \otimes \mathrm{CH}^q_Z(X) \to \mathrm{CH}^{p+q}_{Y \cap Z}(X)_{\mathbb{Q}}$$
>
> *with the following properties:*
>
> (1) *It makes $\bigoplus_Y \mathrm{CH}^*_Y(X)_{\mathbb{Q}}$ a graded commutative ring with unit $[X] \in \mathrm{CH}^0(X)$.*
>
> (2) *It is compatible with change of support maps*
>
> $$\mathrm{CH}^p_Y(X) \to \mathrm{CH}^p_{Y'}(X) \quad \textit{for } Y \subset Y' \subset X.$$
>
> (3) *If Y and Z intersect properly, that is $\operatorname{codim}_X(Y \cap Z) = \operatorname{codim}_X(Y) + \operatorname{codim}_X(Z)$, then the inter-*

section product $[Y].[Z] \in \mathrm{CH}^{m+n}_{Y\cap Z}(X)_{\mathbb{Q}}$ *is given by Serre's* Tor*-formula for intersection multiplicities,*

$$m_x(\mathcal{O}_Y, \mathcal{O}_Z) = \sum_{j\geq 0}(-1)^j \operatorname{length}(\operatorname{Tor}_j^{\mathcal{O}_{X,x}}(\mathcal{O}_{Y,x}, \mathcal{O}_{Z,x})).$$

Proof. The pairing is in fact induced by the cup product on the γ filtration on K-theory (see [GS87, Theorem 8.3]),

$$K_0^Y(X) \otimes K_0^Z(X) \to K_0^{Y\cap Z}(X).$$

□

2.4 Intersection pairing on arithmetic Chow groups

Theorem 2.8. *For $p, q \geq 0$, there is an intersection pairing*

$$\begin{gathered}\widehat{\mathrm{CH}}^p(X) \otimes \widehat{\mathrm{CH}}^q(X) \to \widehat{\mathrm{CH}}^{p+q}(X)_{\mathbb{Q}}\\ \alpha \otimes \beta \mapsto \alpha \cdot \beta.\end{gathered}$$

It turns $\widehat{\mathrm{CH}}^*(X)_{\mathbb{Q}} = \bigoplus_{p\geq 0} \widehat{\mathrm{CH}}^p(X)_{\mathbb{Q}}$ into a graded commutative $\mathbb{Q}$-algebra. Moreover, the maps

$$\zeta : \widehat{\mathrm{CH}}^*(X)_{\mathbb{Q}} \to \mathrm{CH}^*(X)_{\mathbb{Q}}$$

and

$$\omega : \widehat{\mathrm{CH}}^*(X)_{\mathbb{Q}} \to \bigoplus_{p\geq 0} A^{p,p}(X(\mathbb{C}))$$

are both ring homomorphisms with respect to this structure.

Let $\mathrm{CH}^p_{\mathrm{fin}}(X)$ be the union of the groups $\mathrm{CH}^p_Y(X)$ as $Y \subset X$ runs over all closed subsets with empty generic fiber, that is, $Y \cap X_{\mathbb{Q}} = \emptyset$. There is a canonical map

$$\mathrm{CH}^p_Y(X) \to \mathrm{CH}^p_{\mathrm{fin}}(X) \oplus Z^p(X_{\mathbb{Q}}).$$

Proof of Theorem 2.8. If α (resp. β) is the class of (Y, g_Y) (resp. (Z, g_Z)), we let $Y \cap Z \in \mathrm{CH}^{p+q}_{\mathrm{fin}}(X)_{\mathbb{Q}} \oplus Z^{p+q}(X_{\mathbb{Q}})_{\mathbb{Q}}$ be the image of $[Y] \otimes [Z] \in \mathrm{CH}^p_Y(X) \otimes \mathrm{CH}^p_Z(X)$ under the maps

$$\mathrm{CH}^p_Y(X) \otimes \mathrm{CH}^q_Z(X) \to \mathrm{CH}^{p+q}_{Y\cap Z}(X)_{\mathbb{Q}} \to \mathrm{CH}^{p+q}_{\mathrm{fin}}(X) \oplus Z^{p+q}(X_{\mathbb{Q}})_{\mathbb{Q}}.$$

Here, in order for the last map to be well-defined, we are to assume that the restrictions $Y_{\mathbb{Q}}$ and $Z_{\mathbb{Q}}$ of Y and Z to the generic fiber $X_{\mathbb{Q}}$ intersect properly, i.e. the components of $Y_{\mathbb{Q}} \cap Z_{\mathbb{Q}}$ have codimension $p+q$. To do this, one invokes a moving lemma for cycles. While one has Chow's moving lemma for smooth quasi-projective $\mathbb{Q}$-varieties, there is no such result for arithmetic varieties. This is the reason for invoking K-theoretic methods upon tensoring by $\mathbb{Q}$ as in Theorem 2.8 above.

Next we define a Green current for $Y \cap Z$. For this we set

$$\begin{aligned}\omega_Y &= dd^c g_Y + \delta_{Y(\mathbb{C})} \in A^{p,p}(X(\mathbb{C})),\\ \omega_Z &= dd^c g_Z + \delta_{Z(\mathbb{C})} \in A^{p,p}(X(\mathbb{C})),\end{aligned}$$

and let

$$g_Y * g_Z = \delta_{Y(\mathbb{C})} \wedge g_Z + g_Y \wedge \omega_Z \in \widetilde{D}^{p+q-1,p+q-1}(X).$$

This is not a priori well-defined; implicitly on the right-hand side it requires the choice of Green forms representing g_Y and g_Z. However, up to addition of a term $\partial u + \overline{\partial} v$, one can assume that g_Z is L^1 on

$X(\mathbb{C}) \setminus Z(\mathbb{C})$ (more precisely, with "logarithmic singularities", with restriction an L^1-form η on $Y(\mathbb{C}) \setminus Y(\mathbb{C}) \cap Z(\mathbb{C})$.

$$(\delta_{Y(\mathbb{C})} \wedge g_Z)(\omega) = \int_{Y(\mathbb{C})\setminus Y(\mathbb{C})\cap Z(\mathbb{C})} \eta \wedge \omega.$$

Formally, we have

$$\begin{aligned} dd^c(g_Y * g_Z) &= dd^c(\delta_{Y(\mathbb{C})} \wedge g_Z) + dd^c(g_Y \omega_Z) \\ &= \delta_{Y(\mathbb{C})} \wedge dd^c(g_Z) + dd^c(g_Y) \wedge \omega_Z \\ &= \delta_{Y(\mathbb{C})} \wedge (\omega_Z - \delta_{Z(\mathbb{C})}) + (\omega_Y - \delta_{Y(\mathbb{C})}) \wedge \omega_Z \\ &= \omega_Y \wedge \omega_Z - \delta_{Y(\mathbb{C})} \wedge \delta_{Z(\mathbb{C})} \\ &= \omega_Y \wedge \omega_Z - \delta_{Y(\mathbb{C})\cap Z(\mathbb{C})}. \end{aligned}$$

One can show that the $*$-product on Green currents is always commutative, and if $X_{\mathbb{Q}}$ is projective, then the $*$-product on Green currents is associative. $\square$

3 Arithmetic intersection theory in general

We will wrap up by stating the main concepts in arithmetic intersection theory over different arithmetic rings, largely analogous to the previous section.

3.1 Arithmetic varieties

Definition 3.1. An *arithmetic ring* is a triple (A, Σ, F_∞) consisting of an excellent regular Noetherian integral domain A, a finite nonempty set Σ of embeddings $\sigma : A \hookrightarrow \mathbb{C}$, and a conjugate-linear involution of $\mathbb{C}$-algebras, $F_\infty : \mathbb{C}^\Sigma \to \mathbb{C}^\Sigma$, such that the diagram

$$\begin{tikzcd} A \arrow[r, "\delta"] \arrow[d, equal] & \mathbb{C}^\Sigma = \prod_{\sigma\in\Sigma} \mathbb{C}_\sigma \arrow[d, "F_\infty"] \\ A \arrow[r, "\delta"] & \mathbb{C}^\Sigma = \prod_{\sigma\in\Sigma} \mathbb{C}_\sigma \end{tikzcd}$$

commutes.

Example 3.2.

(1) If the field of fractions of A is a number field F, let $\Sigma = \mathrm{Hom}(A, \mathbb{C})$ be the set of all embeddings of A into $\mathbb{C}$ and let F_∞ be the usual Frobenius on $\mathbb{C}^\Sigma \simeq \mathbb{C} \otimes_\mathbb{Q} F$ induced by complex conjugation.

(2) $A = \mathbb{R}$ or any subring of $\mathbb{R}$, Σ is the obvious embedding of A into $\mathbb{C}$, and F_∞ is complex conjugation.

There are subsequent notions of homomorphisms of arithmetic rings. Note that there is a natural homomorphism of arithmetic rings $\mathcal{O}_F \to \mathcal{O}_E$ for each extension E/F of number fields, and $\mathbb{Z}$ is an initial object in the category of arithmetic rings.

Definition 3.3. If (A, Σ, F_∞) is an arithmetic ring, an *arithmetic variety* X over A is a morphism $\pi : X \to S = \mathrm{Spec}(A)$ which is flat of finite type, with the following requirements. We require that the generic fiber X_F be smooth, where $F = \mathrm{Spec}(A)$. If $\sigma \in \Sigma$, we write $X_\sigma = X \otimes_{A,\sigma} \mathbb{C}$; we write $X_\Sigma = \coprod_{\sigma\in\Sigma} X_\sigma = X \otimes_A \mathbb{C}^\Sigma$. Finally, we write $X_\infty = X_\Sigma(\mathbb{C})$, the analytic space associated to the scheme X_Σ. Since X_F is smooth over F, X_∞ is a complex manifold, with a continuous involution induced by the conjugate-linear automorphism F_∞ of $\mathbb{C}^\Sigma$.

3.2 Definition of the arithmetic Chow group

Let X be an arithmetic variety over $A = (A, \Sigma, F_\infty)$ with smooth projective generic fiber, and Y is a codimension p integral subscheme of X, then Y_Σ is a subscheme of X invariant under F_∞. Hence, integration over Y_∞ defines a Dirac current δ_{Y_∞} in $\widetilde{D}^{p,p}(X_\infty)$. This extends to a map

$$Z^p(X) \to \widetilde{D}^{p,p}(X_\infty).$$

The real forms (resp. real currents) in $D^{p,p}(X_\infty)$ are those which satisfy $F_\infty^*\alpha = (-1)^p\alpha$.

Definition 3.4. Let $\widehat{Z}^p(X)$ be the subgroup of $Z^p(X) \oplus \widetilde{D}^{p-1,p-1}(X_\infty)$ consisting of pairs $(Z = \sum n_i[Z_i], g)$ such that g is a Green current for Z, with $(Z_1, g_1) + (Z_2, g_2) = (Z_1 + Z_2, g_1 + g_2)$.

Example 3.5. If $Y \subset X$ is an integral subscheme of codimension $p-1$, let $\pi : \widetilde{Y}_\Sigma \to Y_\Sigma$ be the resolution of singularities of Y_Σ with π proper. For $f \in \kappa(Y_\Sigma)^*$, f restricts to a rational function $\widetilde{f}$ on $\widetilde{Y}_\Sigma$. The function $\log|\widetilde{f}|^2$ is real-valued and L^1 on $\widetilde{Y}_\infty$, defining a current in $D^{0,0}(\widetilde{Y}_\infty)$. If $\widetilde{i} : \widetilde{Y}_\infty$ is the natural map, then $\widetilde{i}_*(\log|\widetilde{f}|^2) \in D^{p-1,p-1}(X)$. This current is independent of the choice of resolution. By the Poincaré–Lelong formula applied to X_∞, we have $dd^c i_*(\log|f|^2) = \delta_{\mathrm{div}(f)}$, and by invariance of Y and f under F_∞, the pair

$$\widehat{\mathrm{div}}(f) = (\mathrm{div}(f), -i_* \log|f|^2)$$

is an element of $\widehat{Z}^p(X)$.

Definition 3.6. Let $\widehat{R}^p(X) \subset \widehat{Z}^p(X)$ be the subgroup generated by elements of the form $\widehat{\mathrm{div}} f$. Then, the *arithmetic Chow groups* of codimension p of X is the quotient

$$\widehat{\mathrm{CH}}^p(X) = \widehat{Z}^p(X)/\widehat{R}^p(X).$$

Theorem 3.7. *We have a five-term exact sequence*

$$\mathrm{CH}^{p,p-1}(X) \xrightarrow{\rho} \widetilde{A}^{p-1,p-1}(X_\mathbb{R}) \xrightarrow{a} \widehat{\mathrm{CH}}^p(X) \xrightarrow{\zeta} \mathrm{CH}^p(X) \to 0.$$

Example 3.8. If $X = \mathcal{O}_K$, is the ring of integers of a number field K, then, putting r_1 and r_2 the number of real and complex places of K and $\mathrm{Cl}(\mathcal{O}_K)$ the class group of K, we have

$$1 \to \mu(K) \to \mathcal{O}_K^\times \xrightarrow{\rho} \mathbb{R}^{r_1+r_2} \to \widehat{\mathrm{Pic}}(\mathrm{Spec}\,\mathcal{O}_K) \to \mathrm{Cl}(\mathcal{O}_K) \to 0,$$

where $\rho = \prod_{\sigma\in\Sigma(r_1,r_2)} \log|\cdot|_\sigma^2$ is essentially the Dirichlet regulator. We can identify

$$\widehat{\mathrm{CH}}^1(\mathrm{Spec}\,\mathcal{O}_K) \simeq K^\times \backslash \mathbb{A}_K^\times / U_K,$$

where $\mathbb{A}^\times$ is the idèle group of K and U_K is its maximal compact subgroup.

3.3 The intersection pairing

Reprising the ideas in Theorem 2.8, let $A = (A, \Sigma, F_\infty)$ be an arithmetic ring with fraction field F.

Theorem 3.9. *Suppose that X is an arithmetic variety over A which is regular and has quasi-projective generic fiber X_F. Then*

(1) *For $p, q \geq 0$, there is a pairing*

$$\begin{gathered} \widehat{\mathrm{CH}}^p(X) \otimes \widehat{\mathrm{CH}}^q(X) \to \widehat{\mathrm{CH}}^{p+q}(X)_\mathbb{Q} \\ \alpha \otimes \beta \mapsto \alpha.\beta \end{gathered} \tag{3.1}$$

which is uniquely determined by the following condition: if α (resp. β) is the class of $([Y], g_Y)$ (resp. $([Z], g_Z)$), then $\alpha\dot{\beta}$ is the class of the element

$$([Y], g_Y).([Z], g_Z) = ([Y].[Z], g_Y * g_Z) \in (Z^{p+q}(X_F) \oplus \mathrm{CH}^{p+q}_{\mathrm{fin}}(X))_\mathbb{Q} \oplus \widetilde{D}^{p+q-1,p+q-1}(X_\infty).$$

(2) *The product* (3.1) *makes* $\widehat{\mathrm{CH}}^*(X)_{\mathbb{Q}}$ *into a commutative, associative ring.*

(3) *If* p *or* $q = 1$, *there is a unique pairing*

$$\widehat{\mathrm{CH}}^p(X) \otimes \widehat{\mathrm{CH}}^q(X) \to \widehat{\mathrm{CH}}^{p+q}(X)$$

which is given by the formula

$$([Y], g_Y).([Z], g_Z) = ([Y].[Z], g_Y * g_Z) \in \widehat{Z}^{p+q}(X).$$

for cycles intersecting properly on X. *This pairing induces the pairing* (3.1) *taking values in* $\widehat{\mathrm{CH}}^{p+q}(X)_{\mathbb{Q}}$. *When* $\alpha, \beta \in \widehat{\mathrm{CH}}^1(X)$ *and* $\gamma \in \widehat{\mathrm{CH}}^q(X)$, $q \geq 0$, *we have*

$$\alpha\gamma = \gamma\alpha \in \widehat{\mathrm{CH}}^{q+1}(X)$$

and

$$\alpha(\beta\gamma) = \beta(\alpha\gamma) \in \widehat{\mathrm{CH}}^{q+1}(X).$$

(4) *The maps* $\zeta : \widehat{\mathrm{CH}}^p(X) \to \mathrm{CH}^p(X)$, $\omega : \widehat{\mathrm{CH}}^p(X) \to A^{p,p}(X(\mathbb{C})$, *are ring homomorphisms*

Let $\widehat{\mathrm{CH}}^p(X)_0$ be the kernel of $\omega : \widehat{\mathrm{CH}}^p(X) \to A^{p,p}(X(\mathbb{C}))$, and let $\mathrm{CH}^*(X)_0$ be the subgroup of $\mathrm{CH}^*(X)$ of cycles homologically equivalent to 0 in the generic fiber X_∞, so ζ induces a surjective map from $\widehat{\mathrm{CH}}^*(X)_0$ to $\mathrm{CH}^*(X)_0$.

Theorem 3.10. *Suppose that* X *is an arithmetic variety over* A *and has* projective *generic fiber* X_F. *Then the* $\widehat{\mathrm{CH}}^*(X)_{\mathbb{Q}}$*-module structure on the ideal* $\widehat{\mathrm{CH}}^*(X)_{0,\mathbb{Q}}$ *is induced by a* $\mathrm{CH}^*(X)_{\mathbb{Q}}$*-module structure, i.e. we have a factorization*

$$\begin{tikzcd} \widehat{\mathrm{CH}}^p(X)_0 \otimes \widehat{\mathrm{CH}}^q(X) \arrow[r] \arrow[d, "\mathrm{Id} \otimes \zeta"] & \widehat{\mathrm{CH}}^{p+q}(X)_{0,\mathbb{Q}} \\ \widehat{\mathrm{CH}}^p(X)_0 \otimes \mathrm{CH}^q(X) \arrow[ur] & \end{tikzcd}$$

Corollary 3.11. *Let* X *be an arithmetic variety which is regular and has projective generic fiber* X_F. *Then the product of* (3.1), *restricted to* $\widehat{\mathrm{CH}}^*(X)_0$, *factors through* $\mathrm{CH}^*(X)_0$, *i.e. we have a well-defined product:*

$$\mathrm{CH}^p(X)_0 \otimes \mathrm{CH}^q(X)_0 \to \widehat{\mathrm{CH}}^{p+q}(X)_0.$$

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